Homotopy Theory of COMPUTABLE SPACES

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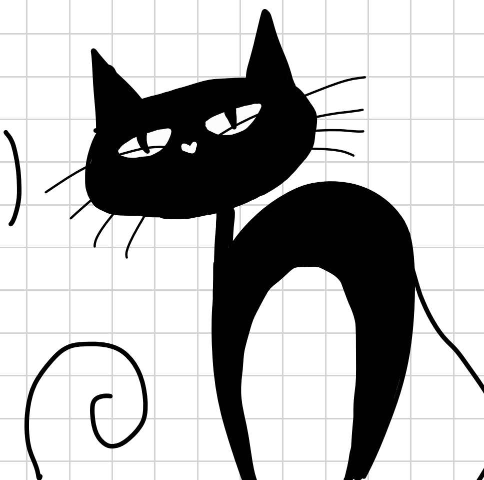
TallCat Seminar

08 April 2025



Calculus of Constructions Propaganda™

Girard Paradox is conflict between

(1) Types = Propositions (instead of Types $\supseteq$ Propositions)

(2) Impredicativity $\dfrac{\Gamma, x:A \vdash B : \mathcal{U}}{\Gamma \vdash \prod_{x:A} B : \mathcal{U}}$ $\left(\text{instead of } \dfrac{\Gamma \vdash A : \mathcal{U} \quad \Gamma, x:A \vdash B : \mathcal{U}}{\Gamma \vdash \prod_{x:A} B : \mathcal{U}}\right)$

kill (2) 	VS	kill (1) 
↓		↓
Martin-Löf Type Theory		Calculus of constructions
$\text{Type}_0 : \text{Type}_1 : \ldots$		Impredicative Prop
$\Downarrow$		$\text{Prop} : \text{Type}_0 : \text{Type}_1 : \ldots$
HoTT...		

Impredicative Universe

- Useful for impredicative encodings of data types

$$\mathbb{N} := \prod_{N:\mathrm{Prop}} N \to (N \to N) \to N \qquad \text{“}\mathbb{N} = \bigcap \{ N \mid N \text{ inductive set} \}\text{”}$$

- But <u>classical</u> set-theoretic models must have $[\![\mathrm{Prop}]\!] \subseteq \{\emptyset, 1\}$
  "Polymorphism is not set-theoretic" - JC Reynolds 1984
- But² <u>constructive</u> set-theoretic models possible
  "Polymorphism is set-theoretic, constructively" – AM Pitts 1987
  ↪ Specifically in Realizability Toposes = "computable sets"
- More recently, interest in impredicative encodings in HoTT

$\Downarrow$

What Semantics?

Homotopy Type Theory (HoTT) Propaganda™

$\vdash A$ Space $\quad\quad \vdash t : A$ point $\quad\quad \vdash p : \mathrm{Id}_A(t_1, t_2)$ path

$\vdash q : \mathrm{Id}_{\mathrm{Id}_A(t_1, t_2)}(p_1, p_2)$ path-between-paths $\quad \ldots \quad$ etc.

$\vdash \mathcal{U}$ Universe	$\vdash A : \mathcal{U}$ Space
$\vdash p : \mathrm{Id}_{\mathcal{U}}(A, B)$ homotopy equivalence ← Univalence axiom	

- Models in categories of spaces such as
    - simplicial sets
    - cubical sets
    - topological spaces
    - anything with a (sufficiently well-behaved) homotopy theory

Frankenstein's Monster

impredicative encodings in HoTT

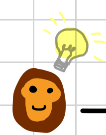
$\Downarrow$

What Semantics?

i.e. what models have on impredicative univalent universe?

Models of Univalence = Spaces

Models of Impredicativity = Computable Sets



Computable Spaces! =

1. a (subcategory of a) realizability topos
2. with a homotopy theory on it
3. with impredicative univalent universe

Plan for the talk

1. Realizability over Scott's Graph Model $=: \mathbb{P}$ (A topological model of computation)
   a) Computation & Topology in $\mathbb{P}$
   b) Realizability Structures over $\mathbb{P}$ (Equilogical Spaces)
   c)* Quotients of Countably-based (QCB) Spaces
2. Structures for Homotopy Theory
   a) Model Categories
   b) Path Categories
3. Homotopy Theory of Equilogical Spaces
   a) Paths are not transitive
   b) Equilogical Spaces is already a homotopy category
   c) Fusing Two Homotopy Theories

1. Realizability over Scott's Graph Model $\mathbb{P}$

1 a) Computation & Topology in $\mathbb{P}$

Defn (Scott's Graph Model)

A Topological Space $\mathbb{P} = (\mathcal{P}\omega, \Omega\mathbb{P})$ where

$\Omega\mathbb{P}$ generated by basic opens $\uparrow x := \{ y \in \mathcal{P}\omega \mid y \supseteq x \}$ for each $x \in \mathcal{P}_{fin}\omega$.

($\Omega\mathbb{P}$ ← Scott Topology) $\diamond$

Fix bijective encoding of pairs & finite sets

$$\langle -, - \rangle : \omega \times \omega \xrightarrow{\cong} \omega \quad \& \quad fin : \omega \xrightarrow{\cong} \mathcal{P}_{fin}\omega$$

and also

$$\langle\!\langle -, - \rangle\!\rangle : \mathbb{P} \times \mathbb{P} \to \mathbb{P} \text{ with } \langle\!\langle x, y \rangle\!\rangle = \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}$$

1 a) Computation & Topology in $\mathbb{P}$

- Any continuous function $f : \mathbb{P} \to \mathbb{P}$ determined by $f|_{P_{fin}\omega}$
- Data of $f|_{P_{fin}\omega}$ may be encoded in

$$\Gamma f := \{ \langle n, m \rangle \in \omega \mid m \in f(\mathrm{fin}(n)) \} \in \mathbb{P}$$

- Any $x \in \mathbb{P}$ encodes a continuous function

$$\Lambda x : y \longmapsto \{ m \in \omega \mid \exists n \in \omega.\ \mathrm{fin}(n) \subseteq y \ \&\ \langle n, m \rangle \in x \}$$

theorem ($\mathbb{P}$ is a model of λ-calculus)

$$\mathrm{Hom}_{\mathrm{Top}}(\mathbb{P}, \mathbb{P}) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbb{P} \quad (\top)$$

$$\Lambda \Gamma = \mathrm{id}$$

$$\forall x \in \mathbb{P}.\ x \subseteq \Gamma \Lambda x$$

□

note: Γ and Λ are themselves continuous.

1 a) Computation & Topology in $\mathbb{P}$

- $\mathbb{P}$ is T_0
- $\mathbb{P}$ has a countable basis $\{\uparrow x \mid x \in \mathcal{P}_{fin}\,\omega\}$

} $\mathbb{P}$ is ωT_0

- $\mathbb{P}$ is a universal ωT_0 space:

theorem (Embedding Theorem)

For X ωT_0,

theorem 1.1.2 "The realizability approach to computable analysis and topology" - A. Bauer

$$\{B : \omega \to \Omega X \text{ subbase enum.}\} \cong \{e : X \hookrightarrow \mathbb{P} \text{ top. embedding}\}$$

$$B \longmapsto e_B(x) := \{n \in \omega \mid x \in B(n)\}$$

□

1b) Realizability Structures over $\mathbb{P}$ – Equilogical Spaces

can put any model of λ-calculus / partial combinatory algebra

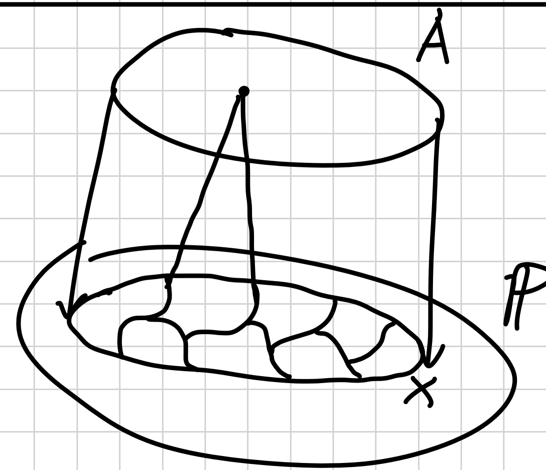
Defn (Modest Set over $\mathbb{P}$)

(A, r) where

(Assembly over $\mathbb{P}$)

- A set
- $r : A \to \mathcal{P}\mathbb{P}$, $a \mapsto r(a)$ realizers
- $r(a)$ non-empty for each $a \in A$
- $a \neq a' \implies r(a) \cap r(a') = \emptyset$

$\wr\wr ?$

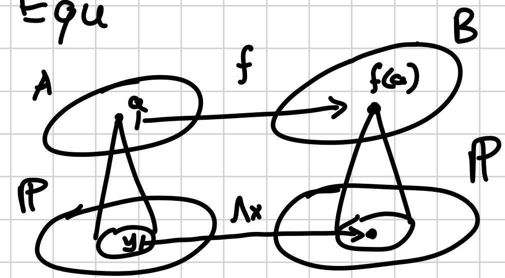


Defn (Equilogical Space)

$(X, \sim)$ where

- X ωT_0 space
- $\sim$ equiv. relation on $|X|$

1b) Realizability Structures over $\mathbb{P}$ – The Category Equ



Defn (Morphism of Modest Sets)

$f : (A, r_A) \longrightarrow (A, r_B)$ where

- $f : A \to B$ function
- $\exists x \in \mathbb{P}. \forall y \in r_A(a). \Lambda x(y) \in r_B(f(a))$

Defn (Morphism of Equilogical Space)

$[f]_\sim : (X, \sim) \longrightarrow (Y, \sim)$ where

- $f : X \to Y$ continuous
- f Equivariant : $\forall x_0, x_1 \in X.\ x_0 \sim x_1 \Rightarrow f(x_0) \sim f(x_1)$
- $f \sim g \overset{\Delta}{\Longleftrightarrow} \forall x \in X.\ f(x) \sim g(x)$
- $[f]_\sim$ equivalence class

1b) Realizability Structures over $\mathbb{P}$ – Realizability Topos $RT(\mathbb{P})$

Defn (Object of $RT(\mathbb{P})$)

$(A, =_A)$ where

- A Set
- $[\bullet =_A \bullet] : A \times A \longrightarrow \wp\mathbb{P}$

"$\wp\mathbb{P}$-valued partial equivalence relation"

- $\exists s \in \mathbb{P}.\ \forall x \in [a=b].\ \Lambda s(x) \in [b=a]$
- $\exists t \in \mathbb{P}.\ \forall x \in [a=b].\ \forall y \in [b=c].\ \Lambda t(\langle\langle x,y\rangle\rangle) \in [a=c]$

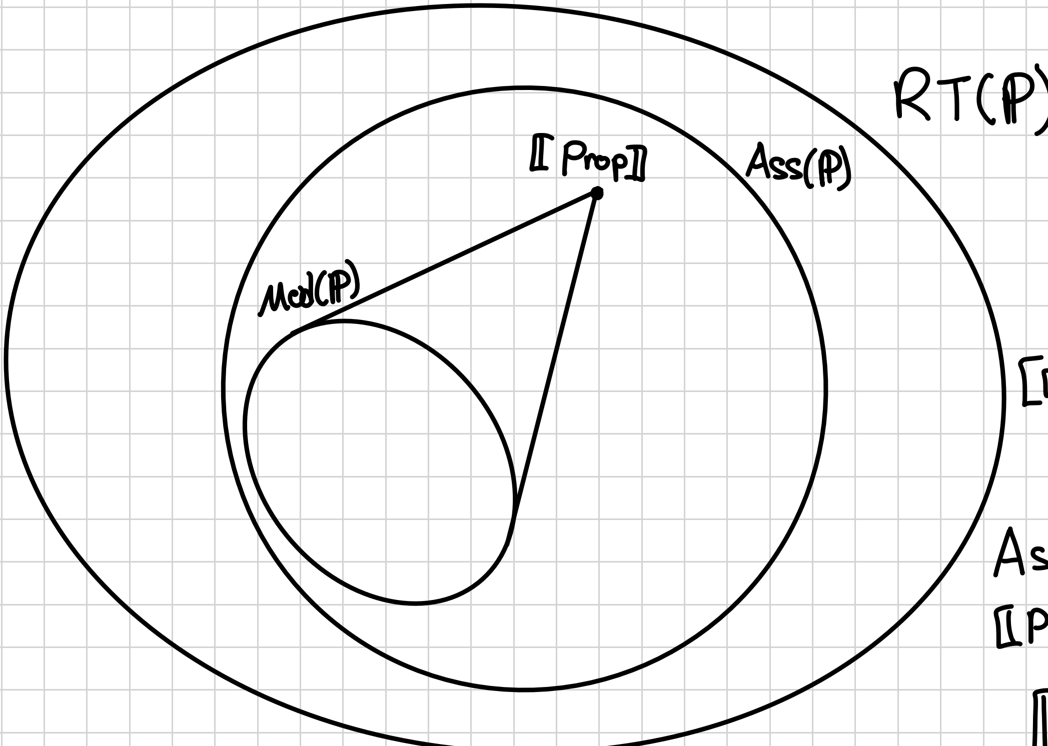
Defn (Morphism of $RT(\mathbb{P})$)

$[F]_= : (A, =_A) \longrightarrow (B, =_B)$ where

"$\wp\mathbb{P}$-valued functional relation"

- $F : A \times B \rightarrow \wp\mathbb{P}$
- $F(a,b) \Rightarrow [a=a] \wedge [b=b]$
- $F(a,b) \wedge F(a,b') \Rightarrow [b=b']$
- $[a'=a] \wedge F(a,b) \wedge [b=b'] \Rightarrow F(a',b')$
- $[a=a] \Rightarrow \bigcup_{b \in B} F(a,b)$
- $[F = F']$ iff $F(a,b) \Rightarrow F'(a,b)$ and $F'(a,b) \Rightarrow F(a,b)$

1b) Realizability Structures over $\mathbb{P}$



$$(X,\sim) \mapsto (X/_\sim,\ r[x]=[x]) \mapsto (X/_\sim,\ =)$$

$$\mathrm{Equ} \to \mathrm{Mod}(\mathbb{P}) \to \mathrm{RT}(\mathbb{P})$$

where

$$\llbracket [x]=[x'] \rrbracket = r[x] \cap r[x'] = \begin{cases} [x] & \text{if } x\sim x' \\ \emptyset & \text{otherwise} \end{cases}$$

$\mathrm{Ass}(\mathbb{P})$ model of CoC with

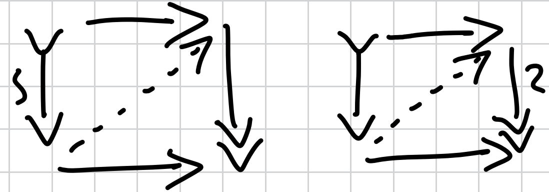
$\llbracket \mathrm{Prop} \rrbracket$ = assembly of modest sets

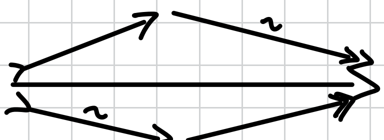
$$\left\llbracket \prod_{a:A} B(a) \right\rrbracket = \bigcap_{a\in\llbracket A\rrbracket} \llbracket B(a) \rrbracket$$

2. Structures for Homotopy Theory

2.a) Model Categories

- Category $\mathcal{C}$ with fin. limits + colimits
- Weak Equivalences $\xrightarrow{\sim}$ Fibrations $\twoheadrightarrow$ Cofibrations $\rightarrowtail$
- ...
- with lifting properties

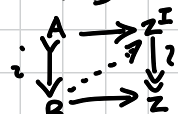


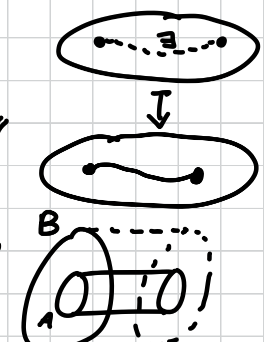
-  factorization

example

- Top with $W = \{\text{homotopy equivalence}\}$

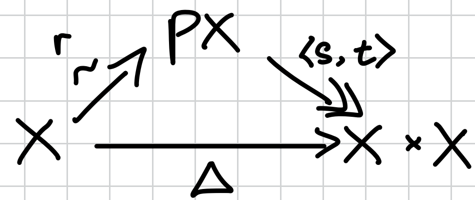
$F = \{ \downarrow p \mid$  $\}$ "cylinder lifting"

$C = \{ \text{closed} \rightarrowtail i \mid$  $\}$ "cylinder extension"



2.b) Path Categories

- Category $\mathcal{C}$
- Two classes of maps $W(\xrightarrow{\sim})$ and $F(\twoheadrightarrow)$
- $\forall X.\ \exists$ <u>Path object</u> PX,



- ...

<u>remark</u> Given interval object $I \in \mathcal{C}$ with good properties, can induce path category by $PX = X^I$ and $W = \{\text{homotopy equivalences}\}$

3. Homotopy Theory of Equilogical Spaces

3a) Paths in Equ are not transitive

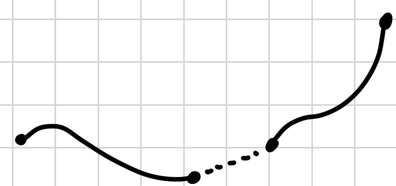
$[0,1]$ wT_0 $\Longrightarrow$ $I = ([0,1], =) \in$ Equ

Defn (?) **homotopy** $H: f \simeq g$ between $f, g: X \to Y$ is

$H: X \times I \to Y$ s.t. $H(-,0) = f$ and $H(-,1) = g$

$f: X \to Y$ **homotopy equivalence** if $\exists g: Y \to X$ s.t. $gf \simeq id$ and $fg \simeq id$

Problem $\simeq$ is not transitive.



Obvious Solution (?) Replace $\simeq$ by transitive closure $\simeq^*$

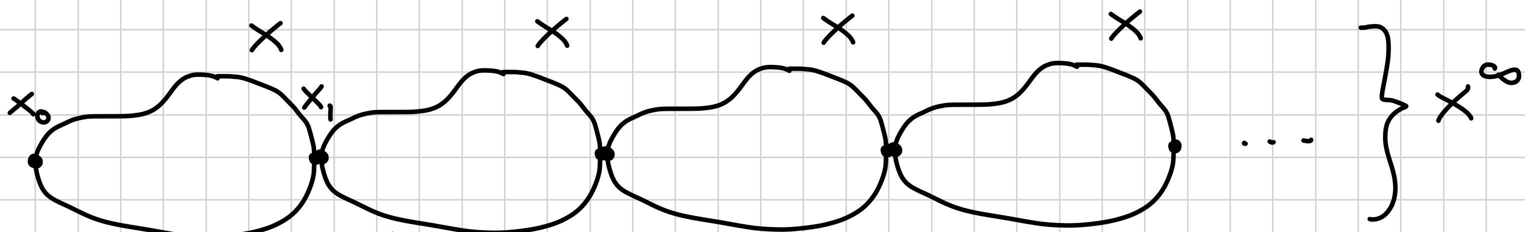
But

Theorem There is no model structure on Equ where W is the set of $\simeq^*$-equivalences.

3a) Paths in Equ are not transitive

Theorem There is no model structure on Equ where W is the set of $\simeq^*$-equivalences.

proof sketch Model structure axioms / abstract nonsense entails existence of space X with $x_0 : 1 \overset{\sim}{\rightarrowtail} X$ and $x_1 \not\simeq x_0$. Glue ∞ copies of X:



$$\begin{array}{ccc} 1 & \overset{x_0}{\underset{\sim}{\rightarrowtail}} & X \\ {\scriptstyle x_1}\downarrow & & \downarrow \ \ulcorner \\ 1 \overset{x_0}{\underset{\sim}{\rightarrowtail}} X & \overset{\sim}{\rightarrowtail} & X^2 \overset{\sim}{\rightarrowtail} X^3 \cdots \overset{\sim}{\rightarrowtail} X^\infty \end{array}$$

still contractible.

But this requires $\simeq^n$ for every n, whereas $\simeq^* = \exists n.\ \simeq^n$ ↯

3b) The Hidden Path - Equ is already a homotopy category

- Isn't $\sim$ in $(X, \sim)$ already a kind of path?
- "equality" in Equ is up to $\sim$
- Gluing adds $\sim$ $\Longleftrightarrow$ Homotopy colimits?
- Therefore:

Proposition

Equ is already a homotopy category of a Path Category Eq? where

- morphisms are equivariant continuous functions (NOT equivalence classes)
- Path Category structure induced by $\mathcal{I} = \bullet \cdots\cdots \bullet$, so

$$PX := X^{\mathcal{I}} = \{(x_0, x_1) \in X^2 \mid x_0 \sim x_1\}$$

3b) The Hidden Path – RT($\mathbb{P}$) is already a homotopy category

"Univalent polymorphism" – Benno van den Berg

↗ "homotopy replacement therapy"

Defn (Object of hRT($\mathbb{P}$))

A Tuple $(A, \alpha, A(-,-), i, s, t)$ where

"locally codiscrete 2-groupoid"
= exactly one 2-cell between any pair of 1-cells

- A set of 0-cells
- $\alpha : A \to \mathbb{P}$ realizer
- $\forall a, b \in A.\ A(a,b) \subseteq \mathbb{P}$ 1-cells
- $i, s, t \in \mathbb{P}$ where i computes identity 1-cell, s computes inverse 1-cells, t computes composition of two 1-cells

Defn (Morphism of hRT($\mathbb{P}$))

$f : (A, \alpha) \to (B, \beta)$ where

"2-functor"

- $f : A \to B$
- $\exists f_0 \in \mathbb{P}.\ \bigwedge f_0(\alpha(a)) = \beta(f(a))$
- $f_{(a,a')} : A(a,a') \to B(fa, fa')$
- $\exists f_1 \in \mathbb{P}.\ f_1$ computes $f_{(a,a')}$

"2-nat. trans."

Defn (Homotopy) $h : f \simeq g : (A,\alpha) \to (B,\beta)$ if $h \in \mathbb{P}$ and $\bigwedge h(\alpha(a)) \in B(fa, ga)$.

3b) The Hidden Path - $RT(\mathbb{P})$ is already a homotopy category

"Univalent polymorphism" - Benno van den Berg

theorem

$RT(\mathbb{P})$ is already a homotopy category of $hRT(\mathbb{P})$ with

Path Object $(P(A,\alpha),\pi)$ where

$$P(A,\alpha) = \{(a,a',\rho) \mid a,a' \in A,\ \rho \in A(a,a')\}$$

$$\pi(a,a',\rho) = \langle\langle \langle\langle a,a' \rangle\rangle, \rho \rangle\rangle$$

$$\begin{array}{ccc} a & \xrightarrow{\rho} & a' \\ m\downarrow & \circlearrowleft & \downarrow n \\ b & \xrightarrow[\sigma]{} & b' \end{array}$$

$$P(A,\alpha)((a,a',\rho),(b,b',\sigma)) = \{\langle\langle m,n \rangle\rangle \mid m \in A(a,b),\ n \in A(a',b')\}$$

proof sketch

Construct $hRT(\mathbb{P}) \longrightarrow RT(\mathbb{P})$

$$(A,\alpha) \longmapsto (A,=)$$

where $[a=b] = \{\langle\langle \langle\langle \alpha(a),\alpha(b) \rangle\rangle, \pi \rangle\rangle \mid \pi \in A(a,b)\}$

$$(\tilde{A},\alpha) \longleftarrow\!\shortmid (A,=)$$

where
- $\tilde{A} = \{(a,p) \mid a \in A,\ p \in [a=a]\}$
- $\tilde{A}((a,p),(b,q)) = [a=b]$
- $\alpha(a,p) = p$

3b) The Hidden Path – RT($\mathbb{P}$) is already a homotopy category

theorem

$$\begin{array}{ccc} \mathrm{Eq2} & \overset{i}{\hookrightarrow} & h\mathrm{RT}(\mathbb{P}) \\ {\scriptstyle ho}\downarrow & & \downarrow{\scriptstyle ho} \\ \mathrm{Equ} & \hookrightarrow & \mathrm{RT}(\mathbb{P}) \end{array}$$

and i preserves + reflects

path category structure

3c) Fusing Two Homotopy Theories

proposition $Eq\ell$ has **another path category structure** induced by $I = [0,1]$.

- Homotopy Theory of $[0,1]$ in Equ is "image of" $(Eq\ell, I)$ under $Ho : (Eq\ell, \mathcal{I}) \longrightarrow Equ$.

- Shifts the study to tandem structure $(Eq\ell, I, \mathcal{I})$
  - ↳ What structure is this, combinatorially?
  - ↳ Suggestion: bisimplicial or bicubical sets which are "locally codiscrete" along one axis?

- Is there a corresponding tandem structure on $hRT(\mathbb{P})$?