Stone Duality for Monads

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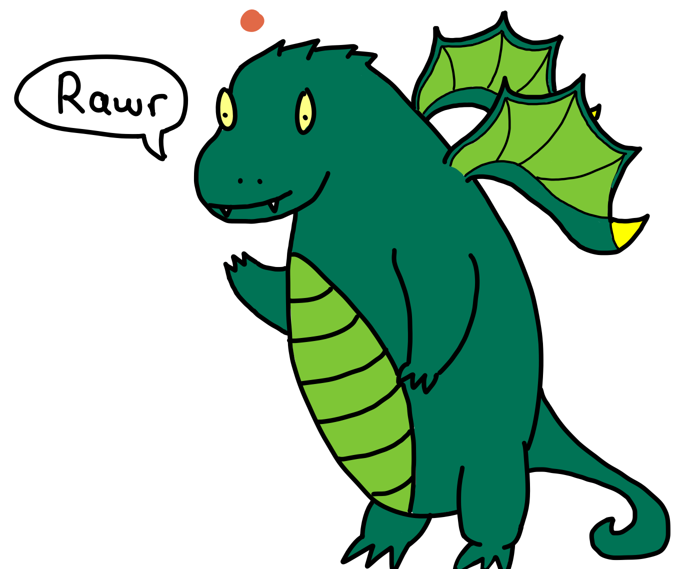

2 June 2026

MFPS @ Ljubljana

Notions of Computation

Plotkin, Power, Moggi :

Notions of Computations = Algebraic Theories

Notions of Computation

Plotkin, Power, Moggi:

Notions of Computations = $\overbrace{\text{Algebraic Theories}}^{}$ $\overset{\text{induce}}{\rightsquigarrow}$ Monads on Set

$$\text{Algebraic Theories} = [\underbrace{\text{operations}}_{\text{Syntax}} \mid \underbrace{\text{equations}}_{\text{Semantics}}]$$

$T : \mathrm{Set} \to \mathrm{Set}$,

$\gg= : TA \times TB^A \to TB$,

$\mathrm{return} : A \to TA$

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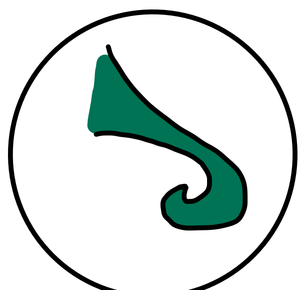
Example Let $2 = \{$  ,  $\}$

HEAD



TAIL

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$\Pi_{(un)flip} = [$ flip/2, unflip /1, unflip /1 $\mid$

return $* =$ flip $\gg\!= \{$  $\mapsto$ unflip ,  $\mapsto$ unflip $\}$

unflip $\gg$ flip = return 

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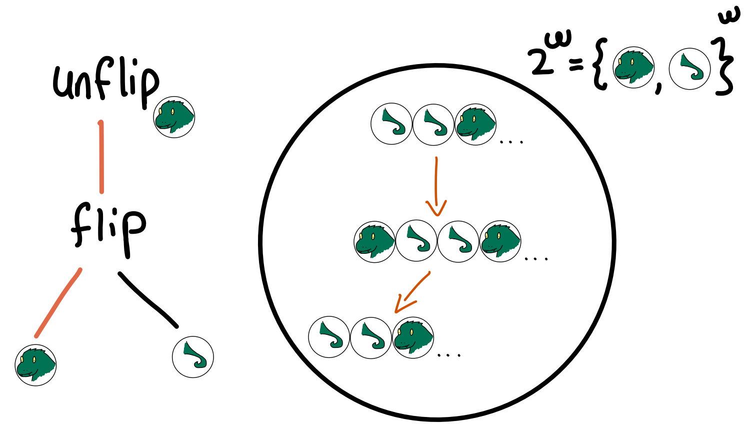
Equational Reasoning comes from Environmental Reasoning

unflip  $\gg$ flip = return  means



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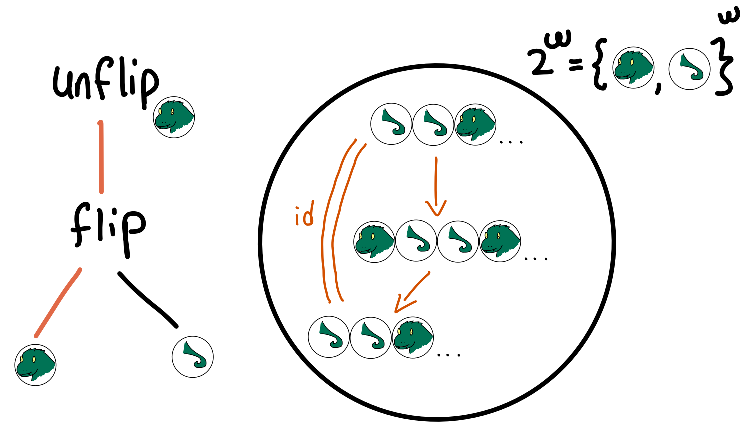

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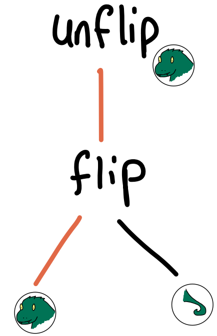

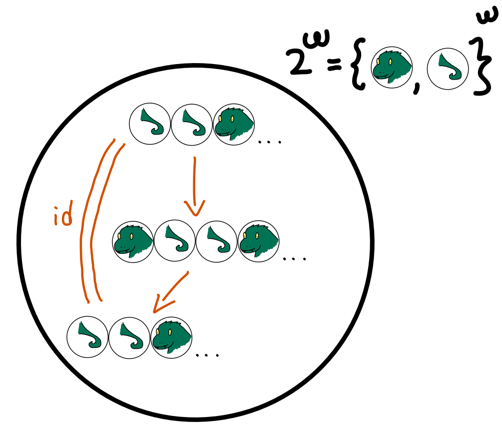

+ Easy to Conjure
+ Incremental Process
− Unintuitive Reasoning

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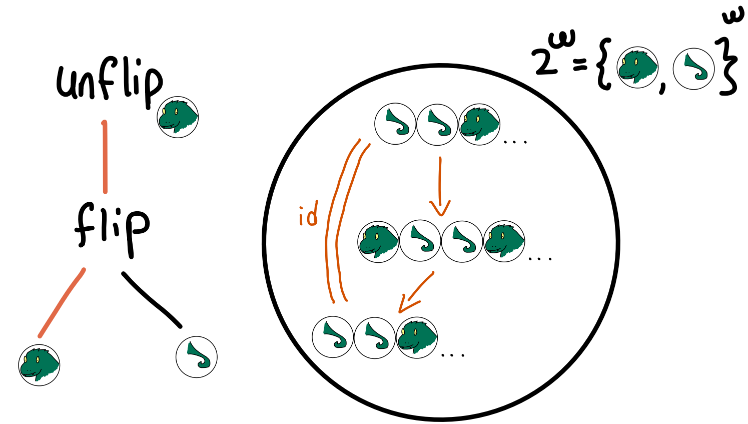

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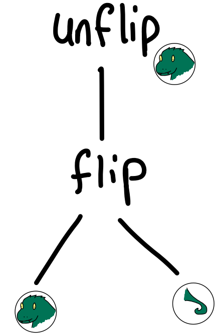

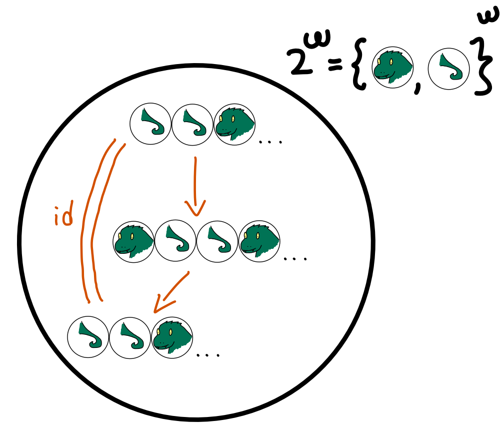

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This is the crux of <u>completeness theorems</u>

Stone Duality (Classically)

$$\begin{tikzcd} \mathrm{BA} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{Top}^{op} \arrow[l, bend left] \end{tikzcd}$$

$$\text{Tests } (B, \wedge, \vee, \neg, \bot, \top) \longmapsto \mathrm{Spec}(B) = \left\{ B \xrightarrow{\beta} \{\text{yes}, \text{no}\} \right\}$$

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Observable Behaviours of Environment States

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$$\mathrm{Clop}(X) \cong \{ X \longrightarrow \{\text{pass}, \text{fail}\} \} \longleftarrow\!\!\shortmid X$$

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\mathrm{BA} \arrow[r, phantom, "\simeq"] \arrow[d, equal] & \mathrm{Stone}^{op} \arrow[d, hook] \\
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$$\begin{tikzcd} \mathsf{Mnd} \arrow[r, bend left, "\mathbb{B}"] \arrow[r, phantom, "\perp"] & \mathsf{TopRetro}^{op} \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$

- Objects = Top-internal categories $\mathbb{C}$, $\mathbb{C}_0$ = states and $\mathbb{C}_1$ = transitions

topological spaces with continous structure maps

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$\mathbb{B}$ehaviour Cat

Γlobal sections

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- Morphisms = retrofunctors

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$$\begin{tikzcd} \mathrm{Mnd} \arrow[rr, bend right, leftarrow, "\Gamma \text{ lobal sections}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{TopRetro}^{op} \arrow[ll, bend right, leftarrow, "\mathbb{B}\text{ehaviour Cat}"'] \end{tikzcd}$$

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$$
\mathbb{C} \overset{R}{\rightsquigarrow} \mathbb{D}
$$

$$
\begin{tikzcd}
\mathbb{C}_0 \arrow[r, "R_0"] & \mathbb{D}_0 \\
& d' \\
c \arrow[r, mapsto] & R_0(c) \arrow[u, "f"]
\end{tikzcd}
$$

Stone Duality (Monadically)

$$\begin{tikzcd}[column sep=huge] \mathrm{Mnd} \arrow[r, bend left, "\mathbb{B}\text{ehaviour Cat}"] \arrow[r, phantom, "\perp"] & \mathrm{TopRetro}^{op} \arrow[l, bend left, "\Gamma\text{lobal sections}"] \end{tikzcd}$$

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$$\begin{tikzcd} c' \arrow[r, mapsto] & d' \\ c \arrow[u, "R_1(f)"] \arrow[r, mapsto] & R_0(c) \arrow[u, "f"'] \end{tikzcd}$$

"functional back simulation"

Stone Duality (Monadically)

$$
\begin{tikzcd}
\mathrm{BA} \arrow[rrr, phantom, "\simeq"] \arrow[dd, "\substack{\text{monad of}\\ \text{if } b \text{ then else}\\ \forall b \in B}"] & & & \mathrm{Stone}^{op} \arrow[dd, "\substack{\text{identity}\\ \text{transitions}\\ \text{only}}"'] \\
& & & \\
\mathrm{Mnd} \arrow[rrr, bend left, "\mathbb{B}\text{ehaviour Cat}"] \arrow[rrr, phantom, "\perp"] & & & \mathrm{TopRetro}^{op} \arrow[lll, bend left, "\Gamma\text{lobal sections}"]
\end{tikzcd}
$$

Top-internal

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$$
\begin{tikzcd}
\mathbb{C} \arrow[r, rightsquigarrow, "R"] & \mathbb{D} \\
\mathbb{C}_0 \arrow[r, "R_0"] & \mathbb{D}_0 \\
c' \arrow[r, mapsto] & d' \\
c \arrow[u, "R_1(f)"] \arrow[r, mapsto] & R_0(c) \arrow[u, "f"'] \arrow[ul, phantom, "\longleftarrow"]
\end{tikzcd}
$$

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Stone Duality (Monadically)

$$
\begin{tikzcd}
\mathrm{BA} \arrow[d, hook, "\text{monad of if } b \text{ then else } \forall b\in B"] \arrow[r, phantom, "\simeq"] & \mathrm{Stone}^{op} \arrow[d, hook', "\text{identity transitions only}"'] \\
\mathrm{HaffUnMnd} \arrow[d, hook] \arrow[r, phantom, "\simeq"] & \mathrm{AmpleTopRetro}^{op} \arrow[d, hook] \\
\mathrm{Mnd} \arrow[r, bend left, "\mathbb{B}\text{ehaviour Cat}"] \arrow[r, phantom, "\perp"] & \mathrm{TopRetro}^{op} \arrow[l, bend left, "\Gamma\text{lobal sections}"]
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\mathrm{HaffUnMnd}_{\omega} \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \mathrm{AmpleTopRetro}^{op} \arrow[d, hook] \\
\mathrm{Mnd}_{\omega} \arrow[r, bend left, "\mathbb{B}\text{ehaviour Cat}"] \arrow[r, phantom, "\perp"] & \mathrm{TopRetro}^{op} \arrow[l, bend left, "\Gamma_{\omega}\text{lobal sections}"]
\end{tikzcd}
$$

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c \arrow[u, "R_1(f)"] \arrow[r, mapsto] & R_0(c) \arrow[u, "f"'] \arrow[ul, phantom, "\longmapsfrom"]
\end{tikzcd}
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Stone Duality (Monadically)

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\mathrm{Mnd}_{\omega} \arrow[rr, bend left, "\mathbb{B}"] \arrow[rr, phantom, "\perp"] & & \mathrm{TopRetro}^{op} \arrow[ll, bend left, "\Gamma_{\omega}"]
\end{tikzcd}$$

Stone Duality (Monadically)

$$\Gamma_\omega \mathbb{C}(A) = \{ s : \mathbb{C}_0 \longrightarrow A' \times \mathbb{C}_1 \mid \qquad \mathrm{dom}(s(c)) = c \}$$

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\mathbf{Mnd}_\omega \arrow[r, bend left, "\mathbb{B}"] \arrow[r, phantom, "\perp"] & \mathbf{TopRetro}^{op} \arrow[l, bend left, "\Gamma_\omega"]
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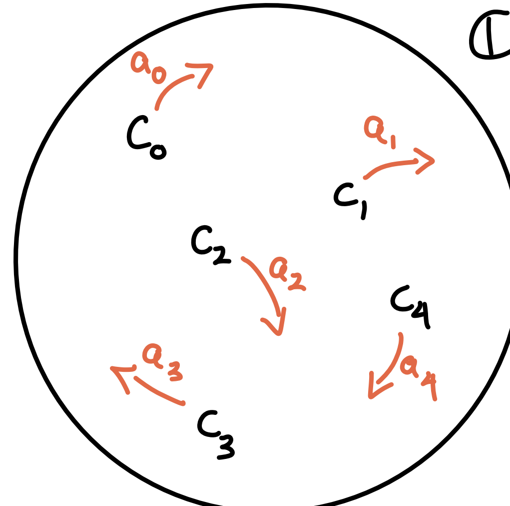
$$\Gamma_\omega \mathbb{C}(A) = \left\{ s : \mathbb{C}_0 \longrightarrow A' \times \mathbb{C}_1 \;\middle|\; A' \subseteq_\omega A \text{ and } \mathrm{dom}(s(c)) = c \right\}$$



"Monad of Brownian Motions"

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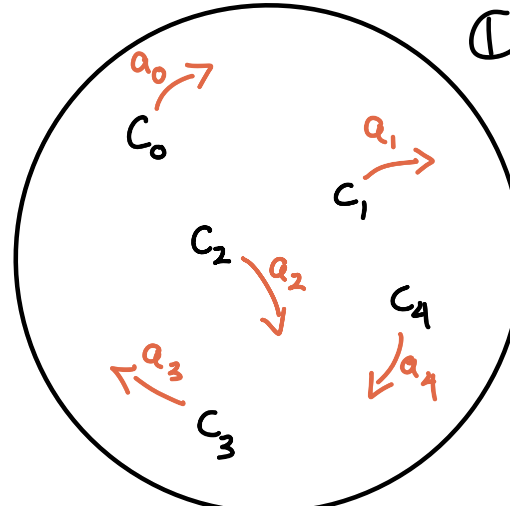
Notice: if $\mathbb{C}_0 = \mathbb{C}_1$ (identity maps only)
then $\Gamma_\omega \mathbb{C}(2) \cong \mathbb{C}_0 \longrightarrow 2$



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The Topological Behaviour Category

From T's perspective,

1. $B_0 T = \{ \beta : T \to \mathrm{Id} \mid \qquad \}$

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(1) $\mathbb{B}_0 T = \left\{ \beta : T \to \mathrm{Id} \;\middle|\; \underset{\forall t \in TA,\ u : A \to TB}{\beta(t \gg= u) = \beta(t \gg u(\beta(t)))} \right\}$

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From T's perspective,

naturality amounts to

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Terminal Top-comodel, for those who know

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2. For $\mathbb{B}_1 T$, from T's perspective, the only transitions that can exist come from $T1$, but subject to

$$\beta \xrightarrow{t \gg\!= v} \beta'' \sim_\beta \beta \xrightarrow{t} \beta' \xrightarrow{v(\beta'(t))} \beta''$$

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Topologized as local homeomorphism corresponding to sheaf generated by global sections $T1$.

The Topological Behaviour Category (Example)

For T_{flip}, $\mathbb{B}_0 T_{flip} \cong 2^{\omega}$ since $\beta(t) = \beta(\underbrace{flip \gg \ldots \gg flip}_{n \text{ times}}) \in 2$, with cantor space topology.

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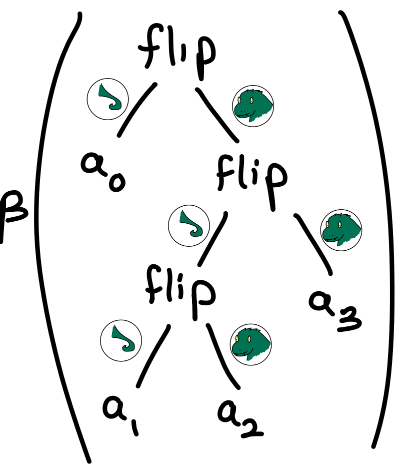

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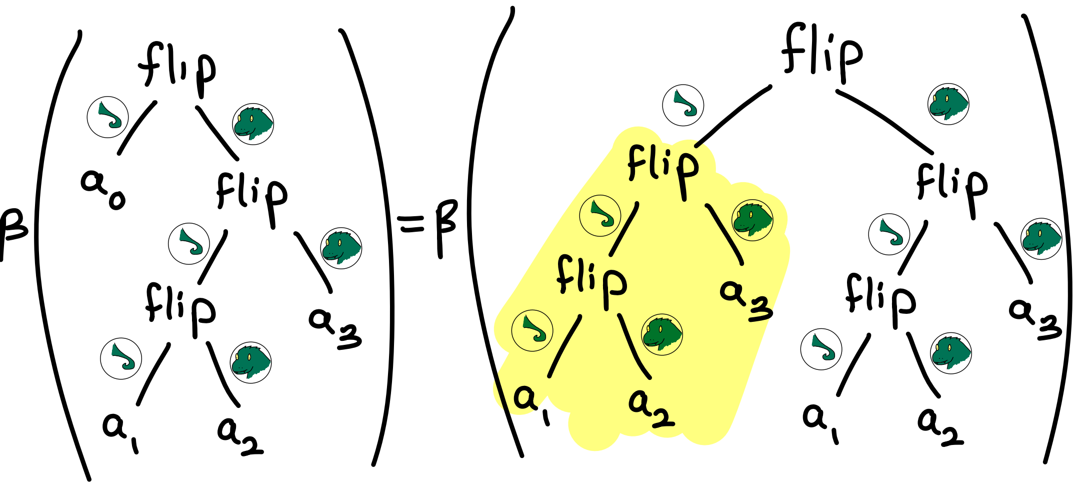

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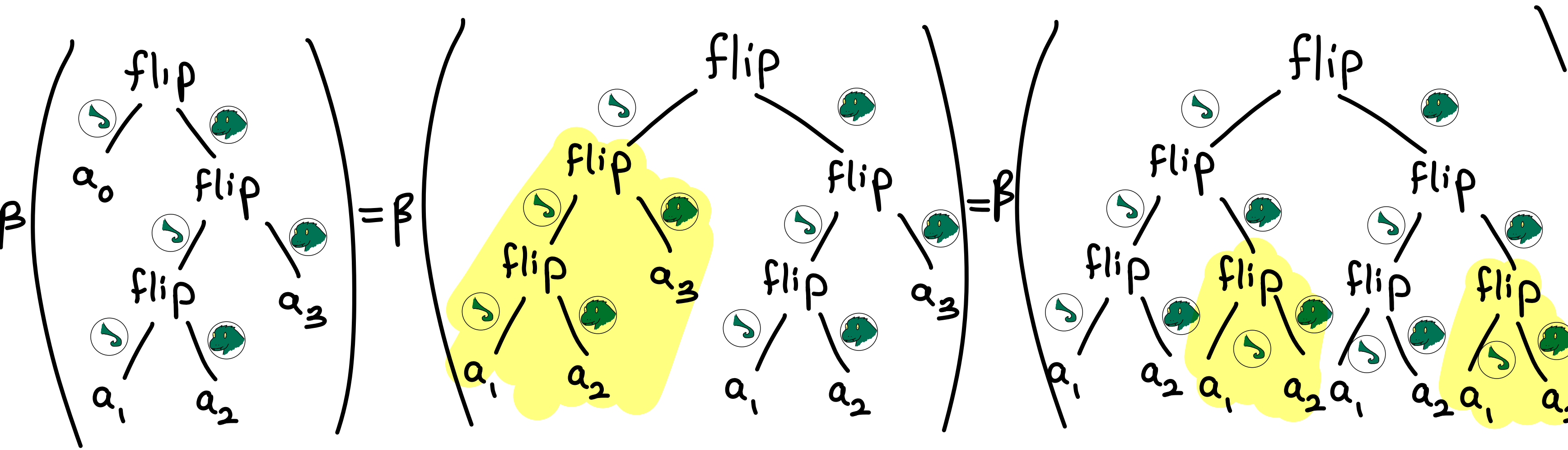

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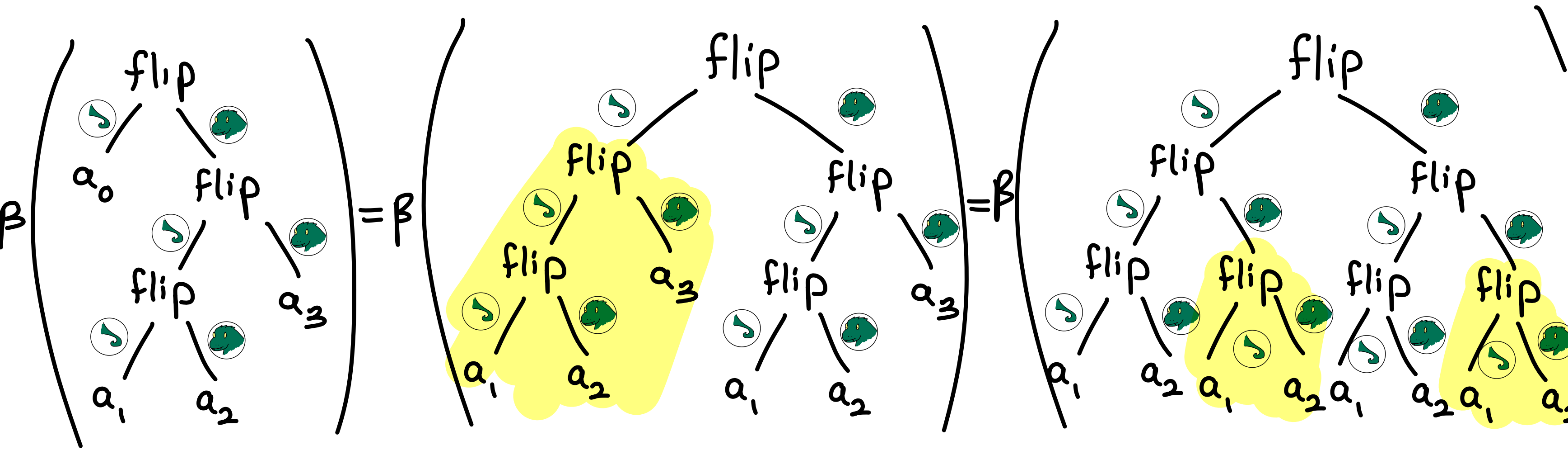

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so $\beta =$  ...

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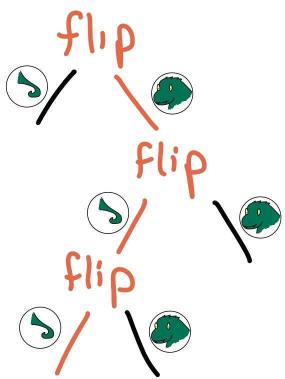

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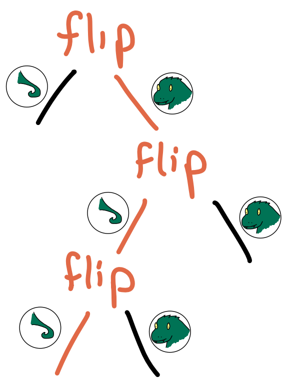

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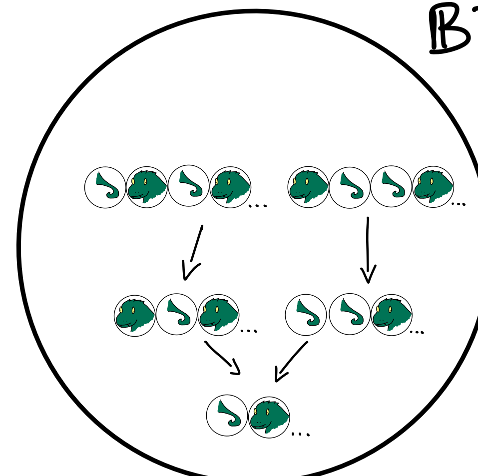
BT



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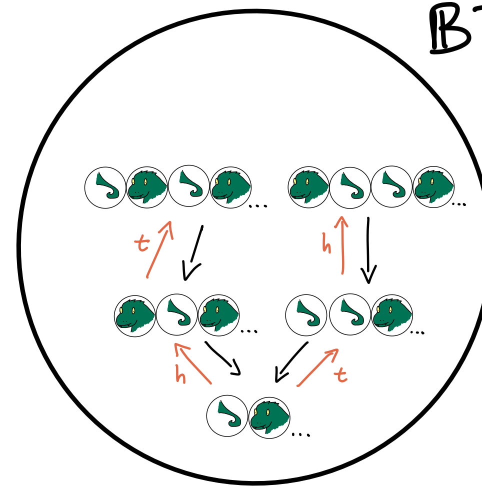
$\mathbb{B}T$



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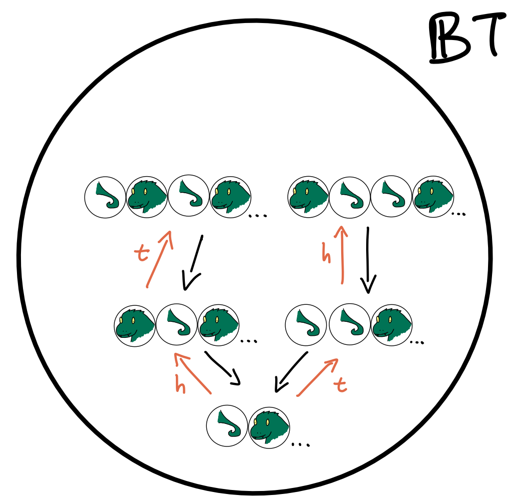
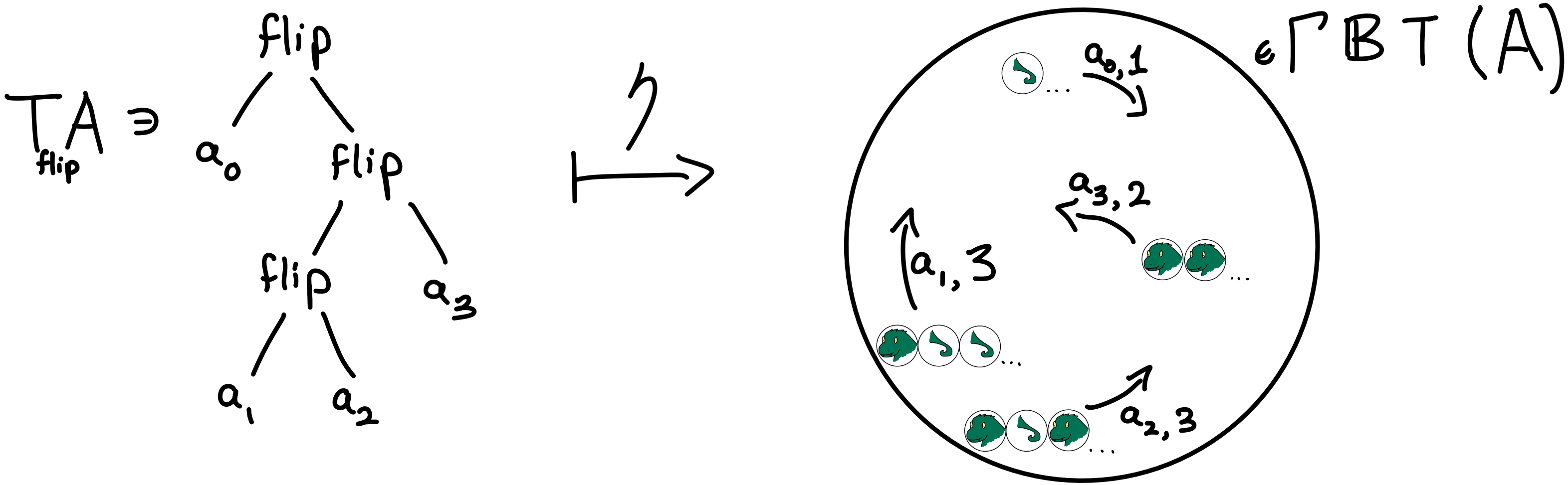
Now is a groupoid!



Prescient Operations

The unit map :



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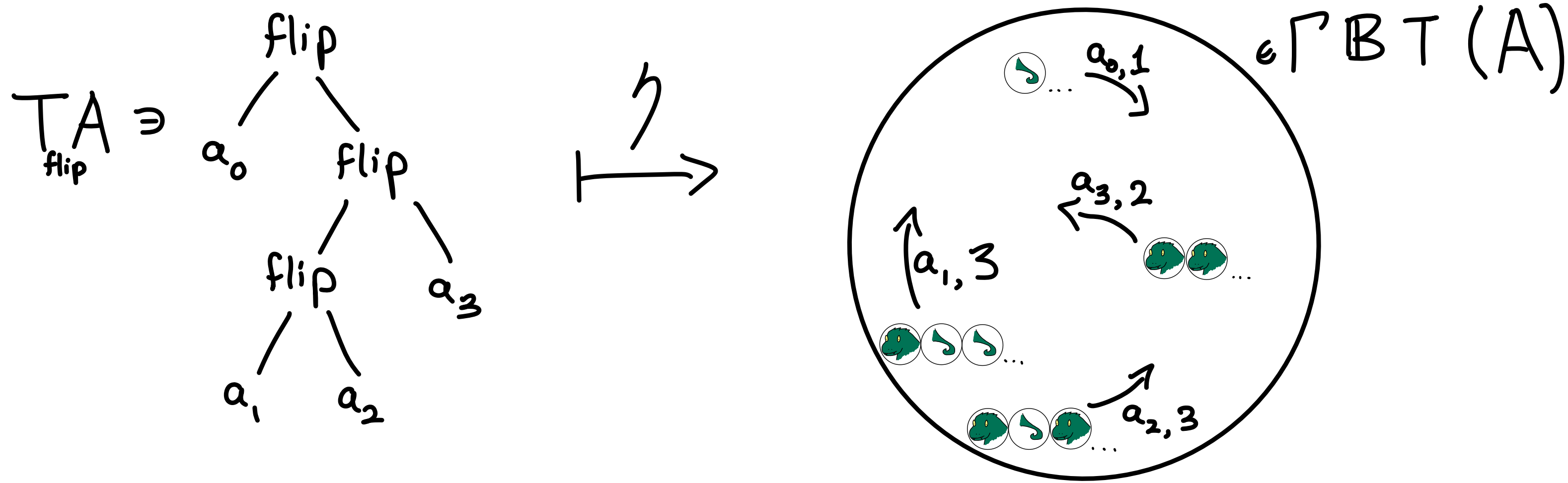

Topology enforces finite information, but not consumption.

Prescient Operations

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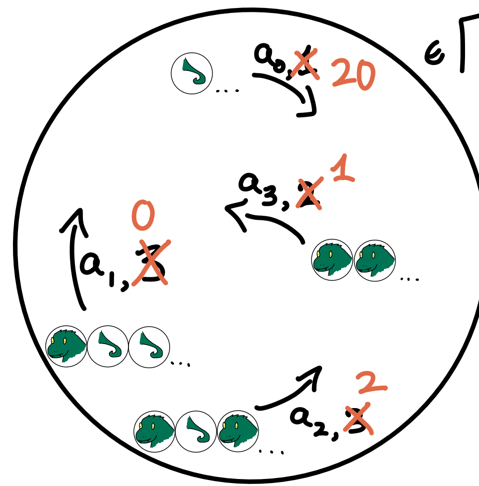
$\nexists\, t \in T_{\text{flip}} A \quad \overset{\eta}{\longmapsto}$



$\in \Gamma \mathbb{B} T (A)$

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$h \gg \text{return } a = \text{return } a$ and $h \gg\!= \lambda a.\, h \gg\!= \lambda b.\, \text{return } (a, b)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

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prop $T \xrightarrow{\eta} \Gamma \mathbb{B} T$ iso iff T hyperaffine-unary.

theorem $\mathrm{HaffUnMnd}_\omega \simeq \mathrm{AmpleTopRetro}^{op}$.

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- Also if T induced by

$$\left[\begin{array}{r|c} \forall x \in \mathbb{R}, & \mathsf{get}_x \gg\!= \lambda n.\mathsf{get}_x \lambda m.\,\mathsf{return}(n,m) = \mathsf{get}_x \lambda n.\,\mathsf{return}(n,n) \qquad \mathsf{get}_x \gg \mathsf{return}\, a = \mathsf{return}\, a \\ \mathsf{get}_x/\mathbb{N} & \mathsf{get}_x \gg\!= \lambda n.\mathsf{get}_y \gg\!= \lambda m.\,\mathsf{return}(n,m) = \mathsf{get}_y \gg\!= \lambda m.\mathsf{get}_x \gg\!= \lambda n.(n,m) \\ & \mathsf{get}_x \gg\!= \lambda n.\mathsf{get}_y \gg\!= \lambda m.\,\mathsf{return}(n,m) = \mathsf{get}_x \gg\!= \lambda n.\mathsf{get}_y \gg\!= \lambda m.f(n,m) \\ & \text{for all } x \neq y \in \mathbb{R} \text{ and } f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \text{ satisfying } \forall n \neq m \in \mathbb{N}. f(n,m) = (n,m) \end{array}\right]$$

Stone Duality (Monadically + Locally)

$$\begin{tikzcd}
\mathbf{BA}_J \arrow[d, hook] \arrow[rr, phantom, "\simeq"] & & \mathbf{StrongZeroDim}^{op} \arrow[d, hook] \\
\mathbf{HaffUnMnd}_{rank} \arrow[d, hook] \arrow[rr, phantom, "\simeq"] & & \mathbf{AmpleLocRetro}^{op} \arrow[d, hook] \\
\mathbf{Mnd}_{rank} \arrow[rr, bend left, "\mathbb{B}"] \arrow[rr, phantom, "\perp"] & & \mathbf{LocRetro}^{op} \arrow[ll, bend left, "\Gamma"]
\end{tikzcd}$$

Grothendieck coverage of partitions

(no ω-restriction)

$\mathbf{LocRetro}^{op}$:

- Objects = categories $\mathbb{C}$, $\mathbb{C}_0$ = states and $\mathbb{C}_1$ = transitions, but $\mathbb{C}_0$ and $\mathbb{C}_1$ are locales and structure maps continuous (i.e. Loc-internal category)

- Morphisms = retrofunctors

$$\mathbb{C} \overset{R}{\rightsquigarrow} \mathbb{D}$$

"$\mathbb{C}_0 \xrightarrow{R_0} \mathbb{D}_0$ functional back simulation"

$$\mathbb{C}_1 \xleftarrow{R_1} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{C}_0$$

in Loc

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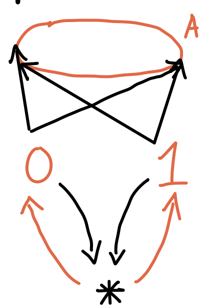
- Stone Spaces from Games:

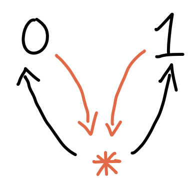
= opponent

= player



$T_{flip} A \cong$ wf winning strategies

Dualize $\rightsquigarrow$



$2^\omega \cong$ winning strategies

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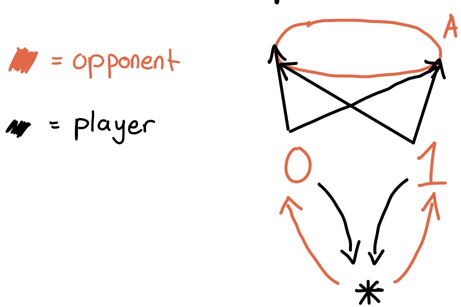

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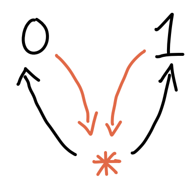

2^ω

$\cong$

winning strategies

Can we tweak this to get quantitative control?

fin.