Let's go to the MALL!

It is colloquial to interpret quantifiers via a 2-player game (e.g. when teaching Kindergarteners the ε-δ definition of limit).

"the sequence $\{x_i\}_{i \in \omega}$ converges to x if no matter what $\varepsilon > 0$ the opponent chooses, you (the player) can find some $\delta \in \omega$ such that any choice of δ by the opponent satisfies

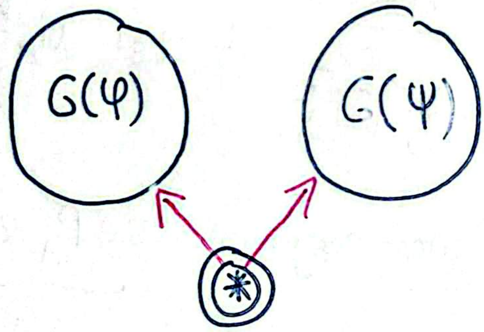
$$\|x_i - x\| < \varepsilon.$$

$$\Updownarrow$$

$$\forall \varepsilon > 0.\ \exists \delta \in \omega.\ \forall i \geq \delta.\ \|x_i - x\| < \varepsilon$$

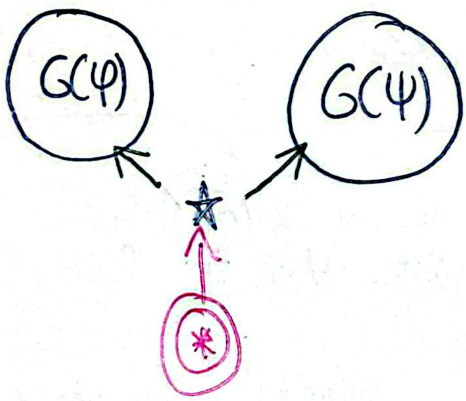
OK, let's take this seriously as a truth definition for first-order logic: Fix a model $(M, \mathbb{I})$

$G(\varphi \wedge \psi) =$



$G(P) =$ if $\mathbb{I}(P)$, if not $\mathbb{I}(P)$.

$G(\varphi \vee \psi) =$



$G(\neg\varphi) = G(\varphi)$

(flip all the move colors)

i.e. $\forall$ sequence of moves $\sigma.a$ $\exists$ a move b s.t. $\sigma.a.b$

$G(\forall x. \varphi(x))$ and $G(\exists x. \varphi(x))$ similar, ranging over $x \in M$.

Define: $(M, \mathbb{I}) \models_G \varphi \iff G(\varphi)$ has a winning strategy.

Hintikka calls this the "semantic game". Ok, and we can actually show that

$$(M, \mathcal{I}) \models \varphi \iff (M, \mathcal{I}) \models_G \varphi$$

just to convince you that this is acceptable.

This is well and good, because the games we play are finite, and this stems from a lack of interiority in the atomic games. But suppose we want to formulate this as a categorical semantics. Then we have to say what is $A \wedge B$, $A \vee B$, $\neg A$ etc. for arbitrary games A and B. And now this gets interesting, because games don't have to be finite. Let's be precise to start with:

<u>Defn</u> A <u>game</u> $A = (M_A, \lambda_A, P_A, W_A)$ consists of

- a set M_A of <u>moves</u>
- $\lambda_A : M_A \to \{\oplus, \ominus\}$

Let $M_A^{\circledast}$ set of alternating finite sequences of moves

- $P_A \subseteq M_A^{\circledast}$ non-empty and $\underbrace{s \in P_A \ \& \ t \sqsubseteq s \Rightarrow t \in P_A}_{\text{prefix-closed}}$.

$\underbrace{\text{set of } \underline{\text{positions}}.}_{}$

Define P_A^{∞} the set of all infinite sequence of moves s s.t. $\forall t \sqsubseteq s$ finite, $t \in P_A$.

- $W_A \subseteq P_A^{\infty}$ winning sequences.

<u>Defn</u> A <u>play</u> of A is $\pi \in P_A^{\leq \infty}$ s.t. $\pi = m^{\ominus} . p$ for some p.
(Opponent starts first, but there are <u>positions</u> starting with Player).

<u>remark</u> Finite plays are winning if $\pi = p . m^{\oplus}$, otherwise losing.

Defn A **strategy** is a non-empty prefix-closed **subset** $\sigma \subseteq P_A$ satisfying

(s1) $a.s \in \sigma \Rightarrow a^{\ominus}$

(s2) $s.a^{\oplus} \in \sigma$ & $s.b^{\oplus} \in \sigma \Rightarrow a = b$

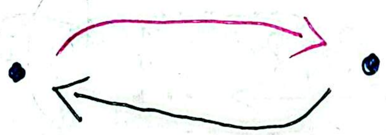
(s3) $s \in \sigma$ & $s.a^{\ominus} \in P_A \Rightarrow s.a^{\ominus} \in \sigma$

σ defines $\hat{\sigma} : \underset{\in \sigma}{s.a^{\ominus}} \longmapsto \underset{\in \sigma}{s.a^{\ominus}.b^{\oplus}}$ (if any)

A **counter-strategy** is $\tau \subseteq P_A$ s.t. (s1) and $(s2)^{\ominus}$ and $(s3)^{\oplus}$

Define $\langle \sigma | \tau \rangle = \bigsqcup (\sigma \cap \tau)$ the play resulting from σ against τ. σ is **winning** if $\langle \sigma | \tau \rangle$ is winning for every τ.

example In practice can make positions finite, to draw pictures like

$L =$ 

formally, $M_L = \{ !, ? \}$, λ_L as coloured,

$P_L = \{ \epsilon, ?, !, ?!, !?, \ldots \}$

Take $W_L = \{ !?!?!?\ldots \}$ (and not the other one)

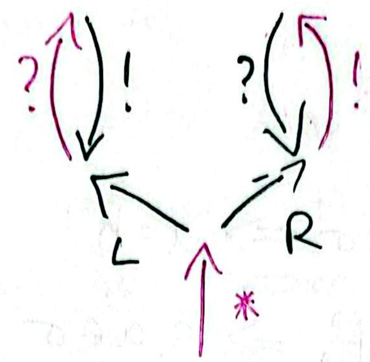
Defn The game $\neg A = (M_A, \overline{\lambda_A}, P_A, P_A^{\infty} - W_A)$. The game $A \vee B$

$M_{A \vee B} = M_A + M_B + \{ *^{\ominus}, L^{\oplus}, R^{\oplus} \}$ $\lambda_{A \vee B}$ as per λ_A and λ_B and as displayed

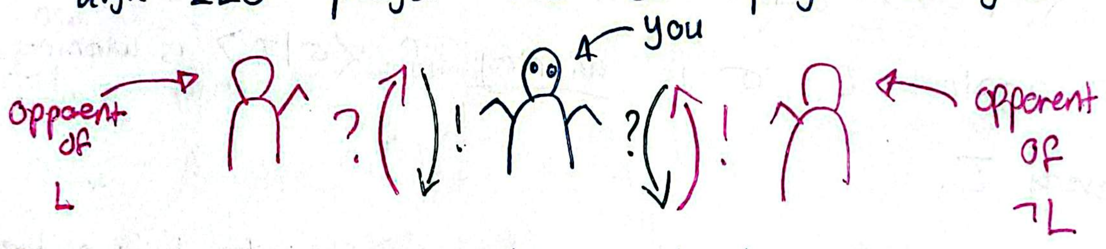
$P_{A \vee B} = \downarrow_{\sqsubseteq}$ of $P_A^{\oplus} + P_B^{\oplus} \cup \{ *.L.P_A^{\ominus} + *.R.P_B^{\ominus} \}$

$W_{A \vee B} =$ —— || —— with W instead of P.

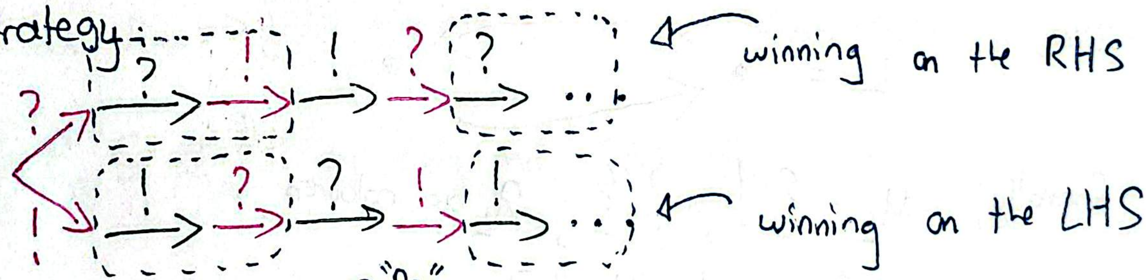
example The game $L \vee \neg L$



There is no winning strategy, i.e. $\nvDash L \vee \neg L$.
RIP the law of excluded middle.
(Some remark about "proof games" of Hintikka)
So the problem is that you had to pick one game to play, either L or $\neg L$. But suppose you are a high ELO player and can play two games at once?



Remember that you only need to win one, so here's the strategy:



Defn The game $A \overset{\text{"Par"}}{\parr} B$:

$M_{A \parr B} = M_A + M_B$, $\lambda_{A \parr B}$ as in λ_A or λ_B.

$$P_{A \parr B} = \{ s \in M_{A \parr B}^{\circledast} \mid s\restriction_A \in P_A \text{ and } s\restriction_B \in P_B \text{ and } \forall G \in \{A,B\}. \ \forall t \sqsubseteq s.\ t = t'.m^{\oplus}.n^{\ominus} \text{ and } m \in M_G \Rightarrow n \in M_G$$

$$W_{A \parr B} = \{ w \in P^{\infty}_{A \parr B} \mid w\restriction_A \in W_A \text{ or } w\restriction_B \in W_B \}$$

theorem $\vDash \neg A \parr A$.

proof Do the "copycat strategy" e_A. Note that

$$W_{\neg A \parr A} = \{ \omega \mid \omega\restriction_{\neg A} \notin W_A \text{ or } \omega\restriction_A \in W_A \}$$

And
$$e_A = \{ s \in P_{\neg A \parr A} \mid \forall t \sqsubseteq s.\ t = t'.m^{\ominus}.n^{\oplus}.$$
$$(m \in M_{\neg A} \Rightarrow n \in M_A \text{ and } m = n)$$
and
$$(m \in M_A \Rightarrow n \in M_{\neg A} \text{ and } m = n)$$

i.e. e_A inductively generated by $\dfrac{}{\epsilon \in e_A}$ $\dfrac{s \in e_A \quad m^{\ominus} \in M_A}{s.m^{\ominus}.m^{\oplus} \in e_A}$ and vice versa.

Then $\forall \tau$ counter-strategy,

$$\omega' = \langle e_A \mid \tau \rangle\restriction_{\neg A} = \langle e_A \mid \tau \rangle \restriction_A$$

So clearly $\omega' \notin W_A$ or $\omega' \in W_A$ (by meta-LEM which is fine, because our meta-logic is justified by finite games)

and hence $\langle e_A \mid \tau \rangle \in W_{\neg A \parr A}$.

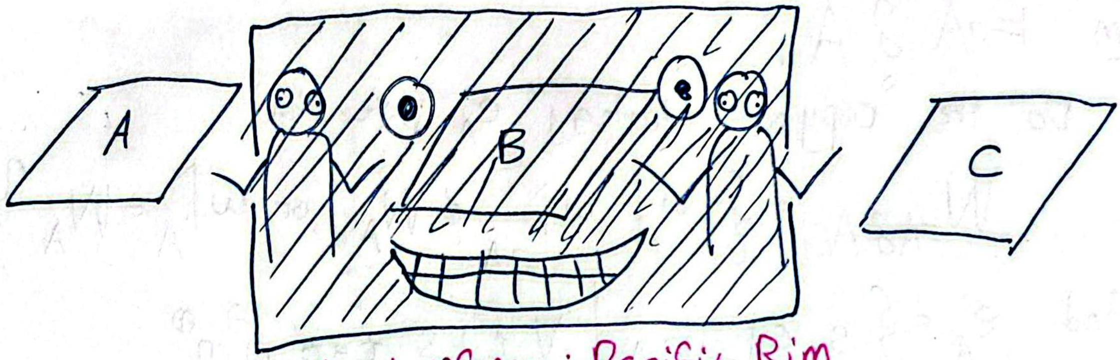
(essentially, one of your opponents is bound to win, so just play them against each other. Notice that this requires draws to never happen) $\square$

In classical logic, $A \to B \cong \neg A \vee B$. So this suggests we define an "implication" $A \multimap B := \neg A \parr B$, which we can use to define morphisms between games:

$$\mathrm{Hom}(A,B) = \{ \sigma \text{ winning strategy for } \neg A \parr B \}$$

Computability/Complexity theoretic interpretation: reduce the problem of beating B to winning A (which the A-player knows how to do)

What is $\sigma_1 ; \sigma_2$?



Cultural reference: Pacific Rim

Important: The robot must "think" to respond, i.e. internal play must occur.

Defn $\mathcal{L}(A_1, A_2, A_3) = \left\{ s \in \left(M_{A_1} + M_{A_2} + M_{A_3}\right)^{\infty} \;\middle|\; \begin{array}{l} \forall t \sqsubseteq s \text{ finite.} \\ t = t'.m.n \; \& \\ m \in M_{A_i} \; \& \; n \in M_{A_j} \\ \Rightarrow |i-j| \leq 1. \end{array} \right\}$

$$\sigma_1 ; \sigma_2 = \left\{ s\restriction_{A,C} \;\middle|\; s \in \mathcal{L}(A,B,C) \;\&\; s\restriction_{A,B} \in \sigma_1 \;\&\; s\restriction_{B,C} \in \sigma_2 \right\}$$

theorem $\sigma_1 ; \sigma_2$ is winning. (in particular, robot cannot think forever)

So you get a category of (AJ) games $\mathbb{G}$. So,

What is MALL? Multiplicative Additive Linear Logic

$⅋$ M. disjunction $\vee$ Add. disjunction (usually $\oplus$) $\neg$ negation (usually $(-)^{\perp}$)

$\otimes$ M. conjunction $\wedge$ Add. conjunction (usually $\&$) $!, ?$ exponential modalities

theorem (AJ)

$\mathbb{G}$ is fully complete for MLL + MIX (every win. strat morphism comes from a proof)

(But for MALL, need truly concurrent games!)

Defn (Sequent Calculus for MALL) Γ, Δ multi-sets.

(think of $\Gamma \vdash \Delta$ as $\otimes\Gamma \multimap \parr\Delta$)

$$\frac{}{A \vdash A}\,(Ax)$$

(compare with usual $\frac{}{\Gamma, A \vdash A, \Delta}$)

e.g. $\frac{}{B, A \vdash A}$ what if your opponent opens in B?

In general, multiple copies in Γ means you are able to leverage more information about that game (e.g. retrying Boss fights in games). Multiple copies in Δ means more trouble for you.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \parr B} \qquad \frac{\Gamma_1, A \vdash \Delta_1 \quad B, \Gamma_2 \vdash \Delta_2}{\Gamma_1, A \parr B, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

this is the game $A \parr B$ for your opponent, so they get to choose the switches. in the worst cases, they may choose to play exclusively in A or exclusively in B. So if you can confidently address both possibilities, then you can prove this.

Dually,

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, A \otimes B, \Delta_2}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B}$$

you better be ready for one possibility.

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}$$

(you better be ready for both)

(note: no contraction & weakening! these collapse the distinction
$\parr$ vs $\oplus$
$\&$ vs $\otimes$)

And the "bug":

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\ (\text{mix})$$

References

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