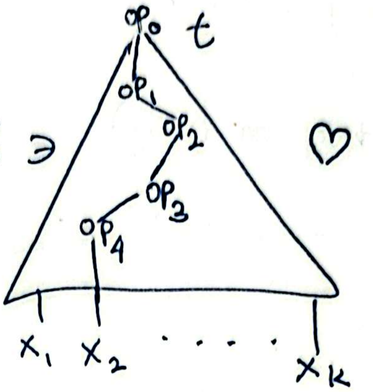
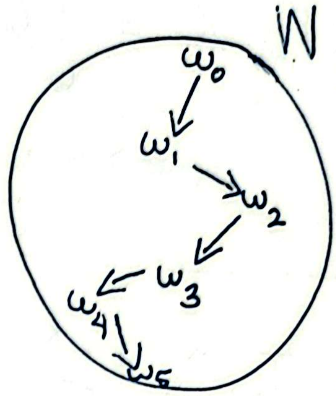
SPECTRUM of MONADS

(T, μ, η) on Set
but
semantics via <u>co</u>algebras

$TX \ni$ 

♡ 

<u>defn</u> a comodel is

$$
\begin{array}{ccc}
W \in \mathrm{CoProd}(Kl(T), \mathrm{Set}) =: \mathrm{Comod}(T) \\
W1 \Downarrow \qquad \Uparrow W\times - \\
\left(W \in \mathrm{Set},\ \rho : \int_X W \times TX \longrightarrow W \times X\right)
\end{array}
$$

write $\rho(w, t) =: (\!|t|\!)(w)$
"co-interpretation"

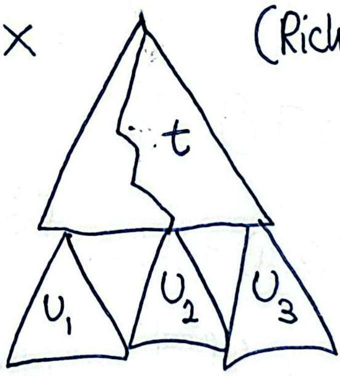
<u>theorem</u> $\mathrm{Comod}(T) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Set}$ (Power, Shkaravska 2004)

so instructive to look at <u>terminal</u> coalgebra $\mathbb{B}_T$.

<u>defn</u> (admissible behaviour) $\beta : \int_X TX \longrightarrow X$ (Richard, 2022)

for $t \in TX$, $u : X \to TY$

$$\beta(\underbrace{t \gg\!= u}_{t(\lambda x.\, u_x)}) = \beta(\underbrace{t \gg u_{\beta(t)}}_{t(\lambda x.\, u_{\beta(t)})})$$



naturality $\iff \beta(t \gg \underbrace{\text{return } x}_{\eta(x)}) = x$

<u>Comodel</u>

$$\mathbb{B}_T \times TX \longrightarrow \mathbb{B}_T \times X$$
$$(\beta, t) \longmapsto (\beta(t \gg -), \beta(t))$$

<u>Terminal</u>

$$W \longrightarrow \mathbb{B}_T$$
$$w \longmapsto \beta_w : t \mapsto (\!|t|\!)(w),$$

examples

1. $T_{inp} X := \{\text{binary trees with leaves in } X\} \rightsquigarrow \beta(\underbrace{b \gg b \gg \ldots \gg b}_{n\text{-times}})$

β determined by

$\iff$ free theory on $b(x,y)$

$\mathbb{B}_{inp} \cong 2^{\omega}$

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2. $T_{out} X = 2^* \times X$

$\iff$ free theory on $U_0(x)$, $U_1(x)$ $\rightsquigarrow$ so $\beta = \int_X 2^* \times X \xrightarrow{\pi_X} X$

every term of form $t \gg \text{return } x$

unique

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3. $T_{state} X = (S \times X)^S \rightsquigarrow$

$s \in S \longmapsto (\beta_s : t \mapsto t(s)_1)$

$\beta(\Delta_s) \longmapsto \beta \in \mathbb{B}_{state}$

$\iff$ generated by $get \in TS$, $\forall s \in S.\ put_s \in T1$.

$\mathbb{B}_{state} \cong S$

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4. $\exists f \in T0 \rightsquigarrow \beta(f) \in 0$ ↯ $\Rightarrow \mathbb{B} \cong \emptyset$

$\exists f(x,y) \in T2.\ f(x,y) = f(y,x) \rightsquigarrow \beta(f(x,y)) = x \Rightarrow$

$y = \beta(f(y,x)) = x$ ↯.

so $\mathbb{B} \cong \emptyset$

(RIP Most typical alg. structures)

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5. For $B \in BA$,

T_B generated by $\forall b \in B.\ b \in T_B 2$. $\rightsquigarrow$ β determined by $\underbrace{\beta|_{T2}}_{\text{ultrafilter}}$

(satisfying some equations)

$\mathbb{B}_B \cong |Spec(B)|$

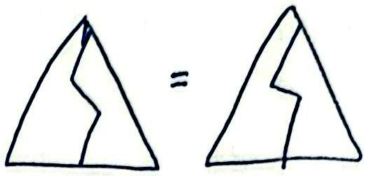
$T_B X \cong \{d : X \to B \mid supp(d) \text{ finite},\ d[supp(d)] \text{ partitions } B\}$

(Richard 2024)

Example (5) suggests $\mathbb{B}_T = |\mathrm{Spec}(T)|$. Here is another indicator.

<u>defn</u> (Behaviour Category)

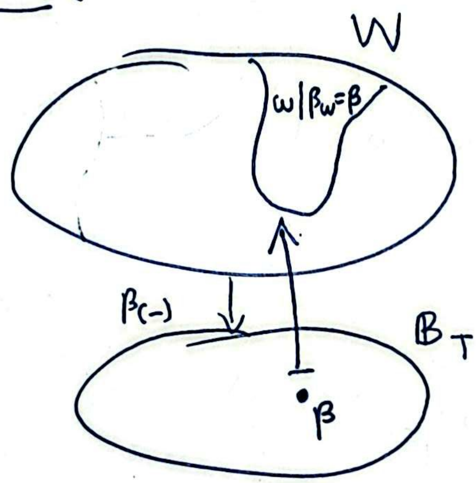
$$\mathrm{Mor}(\mathbb{B}_T) = \sum_{\beta \in \mathbb{B}_T} T1/\sim_\beta \qquad t_1 \sim_\beta t_2 \iff$$



i.e. generated by $t \gg\!= u \sim_\beta t \gg u(\beta(t))$

$$\mathrm{dom}([t]_\beta) := \beta \qquad \mathrm{cod}([t]_\beta) := \beta(t \gg -)$$

<u>theorem</u> (Richard 2022) $\mathrm{Comod}(T) \cong \mathbb{B}_T\text{-Set} \simeq \mathrm{Psh}(\mathbb{B}_T)$



$$\begin{tikzcd} W \arrow[d] & & \mathrm{Mor}(\mathbb{B}_T) \arrow[dll] \arrow[dr] & \\ \mathbb{B}_T & & & \mathbb{B}_T \end{tikzcd}$$

But $\mathrm{Comod}(T) = \mathrm{Coprod}(\mathrm{Kl}(T), \mathrm{Set}) \xhookrightarrow{I} [\mathrm{Kl}(T), \mathrm{Set}] = \mathrm{RMod}(T)$

so calls to mind / suggests this is very restricted form of:

<u>Serre's theorem</u> Let R Noetherian CRing,

$$\{\cancel{\text{Fin-Gen, projective}}\ R\text{-modules}\} \simeq \left\{\begin{array}{l}\cancel{\text{locally free}}\ \text{sheaves of} \\ \text{structure-sheaf modules} \\ \text{of } \cancel{\text{constant finite rank}} \\ \text{on } \mathrm{Spec}(R)\end{array}\right\}$$

I haven't given the topology on $\mathbb{B}_T$ yet, but this suggests W as a presheaf gives the <u>stalks</u>, also $\mathrm{Mor}(\mathbb{B}_T)$.

In what sense is $\mathbb{B}_T$ a sheaf? First, there is a natural topology given by subbasic open sets

$$[t \mapsto x] = \{\beta \mid \beta(t) = x\} \qquad \forall t \in TX, x \in X$$

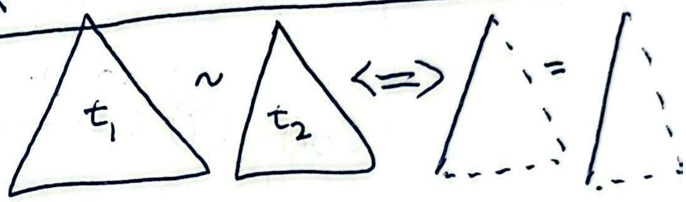
The structure sheaf F_T at first approx, should be a sheaf of monads (just as the ss for a ring is a sheaf of rings) s.t.

$$\operatorname{im}\left(\mathbb{B}_{F_T[t \mapsto x]} \hookrightarrow \mathbb{B}_T\right) = [t \mapsto x]$$

By Duality, we want a quotient $T \twoheadrightarrow F_T[t \mapsto x]$, try

$$t \sim t \gg \text{return } x$$

<u>example</u> $T_{inp}/_{b(x,y) \sim b(x,x)} \implies$



so $\operatorname{Spec}(-\!\!\| \longrightarrow) \cong \{0000....\}$ $\|$

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problem: we allowed congruence, but the condition $[b(x,y) \mapsto x]$ only applies to the first input, so cannot be congruent.

Solution: only allow <u>pre</u> congruence. Then $T/_\sim$ <u>right module</u>

$$\mu/_\sim : T/_\sim T \longrightarrow T/_\sim$$

So $F_T \bigcap_i [t_i \mapsto x_i] = T/_{\bigvee_i \sim_{[t_i \mapsto x_i]}}$

$\Rightarrow$ Stalk over β : $T/_{\bigvee_t \sim_{[t \mapsto \beta(t)]}}$

but

$$\bigvee_t \sim_{[t \mapsto \beta(t)]} = \sim_\beta$$

$\left.\right\}$ $\operatorname{Mar}(\mathbb{B}_T)$ is the total space of germs of $F_T(-)(1)$

What is the right module over β?

$$TX/{\sim_\beta} \cong \underbrace{T1/{\sim_\beta} \times X}_{\text{comodel}}$$

So comodels = "local right modules", and story should apply more generally to right modules.

Diers' spectrum for a multi-adjunction

$$I : \underset{\underset{\text{Coprod}(Kl(T), \text{set})}{\shortparallel}}{\text{Comod}(T)} \hookrightarrow \text{RMod}(T) := [Kl(T), \text{set}]$$

preserves connected limits, so has a left multi-adjoint

$$\sum_{\alpha \in \text{Hom}(M, B_T \times -)} \text{Hom}_{\text{Comod}}(M_\alpha, W \times -) \cong \text{Hom}_{\text{RMod}}(M, W \times -)$$

spec(M)

sanity check

$$\text{Spec}(T) = \text{Hom}(\underset{\overset{\shortparallel ?}{y1}}{T}, B_T \times -) \cong B_T$$

Okay, 1 Is Spec(M) the terminal something?

2 What is M_α?

Another way to view comodels:

$$[T, \mathrm{State}(W)] \cong \int_X TX \to (W \times X)^W$$

If I another set, have right module of State(W)

$$\mathrm{State}(I,W) = (W \times -)^I$$

<u>defn</u> An M-comodel relative to W is a set I with

$$M \to \mathrm{State}(I,W)$$

s.t.

$$\begin{tikzcd} MT \arrow[r] \arrow[d] & \mathrm{State}(I,W)\,\mathrm{State}(W) \arrow[d] \\ M \arrow[r] & \mathrm{State}(I,W) \end{tikzcd}$$

a map $(W,I) \to (W',I')$ is a pair of functions s.t.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \mathrm{State}(I,W) \arrow[d] \\ \mathrm{State}(I',W') \arrow[r] & \mathrm{State}(I,W') \end{tikzcd}$$

<u>prop</u> (Spec(T), spec(M)) is the terminal M-comodel.

$$I \to \mathrm{Spec}(M)$$

$$i \mapsto \alpha_i : m \mapsto$$

That answers 1.

For ②: Category $\mathbb{B}_T$ should be the structure sheaf, and T_β should be the stalks of the structure sheaf (as a sheaf acted on by itself)

<u>defn</u> (operational topology) On $\mathrm{Obj}(\mathbb{B}_T)$, given by (Richard 2023)

$$\forall t \in TX, x \in X.\ [t \mapsto x] := \{ \beta \in \mathbb{B}_T \mid \beta(t) = x \}$$

On $\mathrm{Mor}(\mathbb{B}_T)$, given by

$$\forall m \in T1,\ t \in TX,\ x \in X.\quad [m \mid t \mapsto x] := \{ [m]_\beta \mid \beta(t) = x \}$$

dom is a local homeomorphism, hence a sheaf.

So $T_\beta = T1/\sim_\beta$. Generalizing,

$$M_\alpha = M1/\sim_\alpha \quad \text{where} \quad m \gg= u \sim_\alpha m \gg u(\alpha(m),)$$

We can put operational topology on $\sum_{\alpha \in \mathrm{Spec}(M)} M_\alpha$ and $\mathrm{Spec}(M)$, get "Serre" Theorem:

$$\sum_{\alpha \in \mathrm{Spec}(M)} M_\alpha = \mathrm{Mor}(\mathbb{B}M), \qquad \mathrm{Spec}(M) = \mathbb{B}M$$

$$\begin{tikzcd}
 & \mathrm{Mor}(\mathbb{B}_M) \arrow[dl] \arrow[dr] & & \mathrm{Mor}(\mathbb{B}_T) \arrow[dl] \arrow[dr] & \\
\mathbb{B}_M & & \mathbb{B}_T & & \mathbb{B}_T
\end{tikzcd}$$

If M comodel, $\mathrm{Spec}(M)$ singleton $\{\alpha\}$ and $\mathrm{Mor}(\mathbb{B}_M) = M1/\sim_\alpha = M1 =: W$ to recover the previous picture (is this true for any bimodule?)

If T the monad, this posthoc explains why $\mathbb{B}_T$ is a category.

Examples

We think of right modules in terms of generators and relations

defn $\overset{X}{\sim} \subseteq MX \times MX$ a pre-congruence if

$$\frac{m_1 \overset{X}{\sim} m_2 \qquad t : X \to TY}{m_1 \gg\!= t \overset{Y}{\sim} m_2 \gg\!= t}$$

Given a set of generators, the free right module consists of

$MX = \{(m,t) \mid t : ar(m) \to TX\}$

or for an endofunctor F, FT.

Let $\sim_{[m \mapsto x]}$ generated by $m \sim m \gg$ return x.

- $T_{inp}/\sim_{[b(x,y) \mapsto y]}$ $\quad \begin{array}{c} b \\ / \quad \backslash \\ t_1 \qquad t_2 \end{array} \sim \begin{array}{c} b \\ / \quad \backslash \\ t_3 \qquad t_4 \end{array} \iff t_2 = t_4$

$$T_1 := T_{inp}X/_\sim = \left\{ \text{leaf}(x)\ \forall x \in X,\ \begin{array}{c} b \\ / \quad \backslash \\ \bullet \qquad t \end{array} \forall t \in TX \right\} \cong X + TX$$

with right action

$$\text{leaf}(\text{leaf}(x)) \longmapsto \text{leaf}(x)$$

$$\text{leaf}\left(\begin{array}{c} b \\ / \quad \backslash \\ t_1 \qquad t_2 \end{array} \right) \longmapsto \begin{array}{c} b \\ / \quad \backslash \\ \bullet \qquad t_2 \end{array}$$

$$\begin{array}{c} b \\ / \quad \backslash \\ \bullet \qquad t \in TTX \end{array} \longmapsto \begin{array}{c} b \\ / \quad \backslash \\ \bullet \qquad \mu(t) \end{array}$$

$$\text{Spec}(T_{inp}/\sim) \hookrightarrow \text{Spec}(T)$$

$$\text{Spec}(T_{inp}/\sim) = \{\beta \mid \beta(b(x,y)) = y\} = \{\beta \in 2^\omega \mid \text{head}(\beta) = 1\} \cong 2^\omega.$$

• We can glue two copies of T along T_1:

$$
\begin{tikzcd}
 & T_1 & \\
T \arrow[ur, two heads] & & T \arrow[ul, two heads] \\
 & T_3 \arrow[ul] \arrow[ur] \arrow[u, phantom, "\wedge"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Spec}(T_1) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
\mathrm{Spec}(T) \arrow[dr] & & \mathrm{Spec}(T) \arrow[dl] \\
 & \mathrm{Spec}(T_3) \arrow[u, phantom, "\wedge"] &
\end{tikzcd}
$$

$$T_3 X = \{ \underbrace{(\mathrm{ret}\ x, \mathrm{ret}\ x)\ \forall x}_{\mathrm{return}\ x},\ \underbrace{(b(t_1,t), b(t_2,t))}_{bb(t_1,t_2,t)}\ \forall t_1,t_2,t \in TX \}$$

so T_3 has generators $\mathrm{return} \in T_3 1$ and $bb \in T_3 3$, satisfying

$$(\mathrm{return},\ b(x,y)) \sim (bb,\ (x,x,y))$$

$$\mathrm{Spec}(T_3) = 2^\omega + 2^\omega/\sim \quad \text{where } \mathrm{inl}(\beta) \sim \mathrm{inr}(\beta) \iff \mathrm{head}(\beta) = 1.$$

$$\cong \{ \mathrm{inl}(\beta), \mathrm{inr}(\beta) \mid \mathrm{head}(\beta) = 0 \} + \{ \beta \mid \mathrm{head}(\beta) = 1 \}$$

$$\cong 2^\omega + 2^\omega + 2^\omega$$

$$\cong 3 \times 2^\omega$$

$$(\!| \mathrm{return},\ b(x,y) |\!)(\alpha) = \begin{cases} x & \mathrm{head}\ \alpha = 0 \text{ or } 1 \\ y & \mathrm{head}\ \alpha = 2 \end{cases}$$