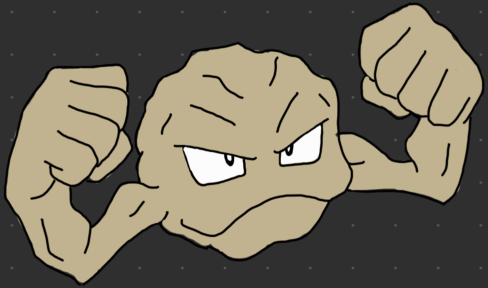
Stone Duality for Monads



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12 December 2025
Birmingham Theory Seminar

This talk in two slides

"Notions of computation $\subseteq$ Monads"

– Eugenio Moggi 1991

↓

Monads $\subseteq$ Notions of $\underbrace{\text{computation}}_{\text{"Interaction with some resource"}}$??

Q: $\forall T$ Monad, $T \cong \mathrm{Comp}(\mathrm{Res}(T))$?

__example__ For State Monad, $\mathrm{Res}((S\times -)^S) = S$ (intuitively)

A: Not quite, but we get an adjunction!

This talk in two slides

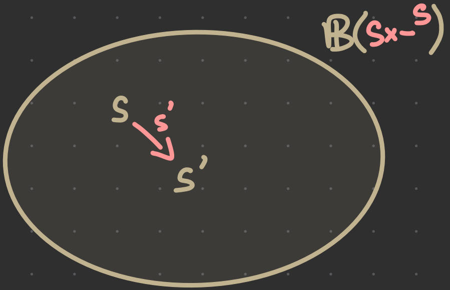
$$\begin{tikzcd}
\mathrm{BoolAlg}^{op} \arrow[r, bend left, "\mathrm{Spec}"] \arrow[r, phantom, "\cong"] \arrow[d, "\text{if-then-else programs}"] & \mathrm{Stone\,Sp} \arrow[l, bend left, "\mathrm{Clop}"] \arrow[d, "\text{Identity Morphisms Only}"'] \\
\mathrm{Mnd}_{fin}(\mathrm{Set})^{op} \arrow[r, bend left, "\mathrm{Res} = \mathbb{B}(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Top\,Cof} \arrow[l, bend left, "\mathrm{Comp} = \Gamma(-)"]
\end{tikzcd}$$

- Objects are Topological Categories
- Morphisms are "cofunctors"

Behavior Category $\mathbb{B}(T)$

- Think of objects of $\mathbb{B}(T)$ as states of $\mathrm{Res}(T)$
- Morphisms as transitions

e.g.



Plan for the Talk

1 Monads & Resources

2 The Terminal Resource as a Category $\mathbb{B}(-)$

3 Computations as global sections $\Gamma(-)$

4 The relationship with Ring Spectra (according to Diers)

A The relationship with Ring Spectra (according to Cole)

1 Monads & Resources

$$\begin{tikzcd}
\mathbf{BoolAlg}^{op} \arrow[rr, bend left, "\mathrm{Spec}"] \arrow[rr, phantom, "\cong"] \arrow[d, hook, "\text{if-then-else programs}"] & & \mathbf{StoneSp} \arrow[ll, bend left, "\mathrm{Clop}"] \arrow[d, hook, "\text{Identity Morphisms Only}"'] \\
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\end{tikzcd}$$

Defn A **monad** $(T, \mathbin{>\!\!>\!\!=}, \mathrm{pure})$ consists of

“bind”

- $T : \mathrm{Set} \to \mathrm{Set}$
- $\mathbin{>\!\!>\!\!=} : TA \times TB^A \to TB \cong \mathrm{Hom}(A, TB) \to \mathrm{Hom}(TA, TB)$
- $\mathrm{pure} : A \to TA$

s.t.

$$t \mathbin{>\!\!>\!\!=} \mathrm{pure} = t \qquad \mathrm{pure}\ a \mathbin{>\!\!>\!\!=} u = u(a)$$

$$(t \mathbin{>\!\!>\!\!=} u) \mathbin{>\!\!>\!\!=} v = t \mathbin{>\!\!>\!\!=} (\lambda a.\ u(a) \mathbin{>\!\!>\!\!=} v)$$

Notation $t_1 \gg t_2$ means $t_1 \mathbin{>\!\!>\!\!=} \lambda\_.\ t_2$

“skip”

Examples

- **State**: $(S \times -)^S$
- **Coin-flipping**: T_{bin} = {binary trees with unlabelled Nodes & leaves in A} = { pure a,

, … }

- **Non-determinism**: $\mathcal{P}_{fin}$
- **Error-throwing**: $- + E$

} Negative Examples

- If B Boolean Algebra, T_B monad of “if-then-else programs”

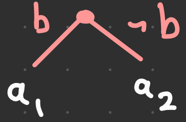
If - Then - Else Programs

$$T_B A = \left\{ A \xrightarrow{t} B \;\middle|\; \begin{matrix} |\{a \in A \mid t(a) \neq \bot\}| = |\mathrm{supp}(t)| < \infty & \forall a \neq a'.\ t(a) \wedge t(a') = \bot \\ t|_{\mathrm{supp}(t)} \text{ injective} & \bigvee_{a \in A} t(a) = \top \end{matrix} \right\}$$

example

if b then a_1 else a_2

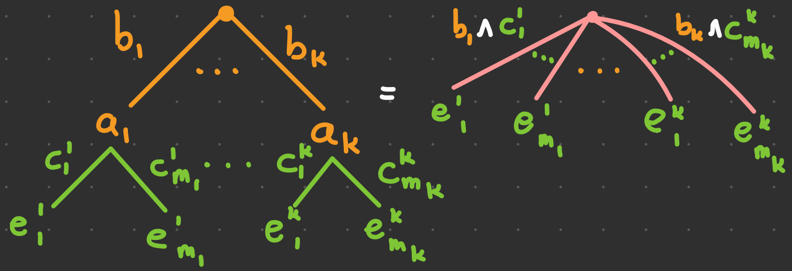
$$b(a_1, a_2) := \lambda a. \begin{cases} b & a = a_1 \\ \neg b & a = a_2 \\ \bot & \text{otherwise} \end{cases} \in T_B A =$$



$$P(a_1 \ldots a_k) := \lambda a. \begin{cases} b_i & a = a_i \\ \bot & \text{otherwise} \end{cases} \in T_B A \quad \text{where } P = \{b_1, b_2 \ldots b_k\} \text{ partitions } B$$

match{ case $b_1 \mapsto a_{b_1}$; case $b_2 \mapsto a_{b_2}$; ... }

$$t \gg= u$$



$$\text{pure } a_0 := \lambda a. \begin{cases} \top & a = a_0 \\ \bot & a \neq a_0 \end{cases}$$

Resources

Defn A comodel of T is $(W, \langle - \rangle)$

$$\forall t \in TA. \quad \langle t \rangle : W \longrightarrow W \times A$$

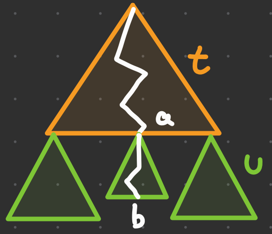
start state

(end state, result)

s.t.

$$\langle \text{pure } a \rangle = W \xrightarrow{\nu_a} W \times A$$

$$\begin{tikzcd} W \arrow[r, "\langle t \rangle"] \arrow[dr, "\langle t \gg= u \rangle"'] & W \times A \arrow[d, "\langle u(a) \rangle_{a \in A}"] \\ & W \times B \end{tikzcd}$$



Examples

- Comodel for T_{Bin} is 2^ω.

$$\left\langle \begin{smallmatrix} & \wedge & \\ t_1 & & t_2 \end{smallmatrix} \right\rangle : \beta \in 2^\omega \longmapsto \langle t_{\text{head } \beta} \rangle (\text{tail } \beta)$$

- Comodel for $(S \times -)^S$ is S. $\langle t \rangle = t : S \longrightarrow S \times A$
- Comodel for $- + E$ is … $\emptyset$, forced by $\langle e \rangle : W \to W \times \emptyset \cong \emptyset$
- Comodel for $\mathcal{P}_{fin}$ is $\emptyset$, $\langle \{0,1\} \rangle = \langle \{1,0\} \rangle = \langle \{0,1\} \gg= \lambda a.\ \text{pure } 1-a \rangle$

Resources

Defn A comodel of T is $(W, \langle - \rangle)$

$\forall\, t \in TA.\quad \langle t \rangle : W \longrightarrow W \times A$

Start State

(end state, result)

s.t.

$\langle \text{pure } a \rangle = W \xrightarrow{\nu_a} W \times A$

$$\begin{tikzcd} W \arrow[r, "\langle t \rangle"] \arrow[dr, "\langle t \gg= u \rangle"'] & W \times A \arrow[d, "\langle u(a) \rangle_{a \in A}"] \\ & W \times B \end{tikzcd}$$

Topological comodel if W top. space & $\langle t \rangle$ continuous.

Examples

Cantor Space

- For T_{Bin}, 2^{ω} with basic opens $[b] = \{ \beta \mid b \text{ prefix } \beta \}\ \forall b \in 2^{<\omega}$
- For $(S \times -)^S$, S with every subset $U \subseteq S$ open.
- For T_B, the Stone space $\mathrm{Spec}(B)$ (will see more soon!)

2 The Terminal Comodel as a Category

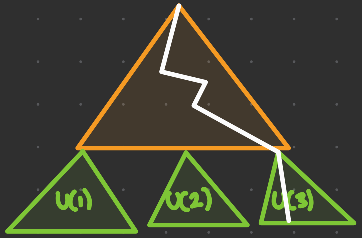
$$\begin{tikzcd}
\mathrm{BoolAlg}^{op} \arrow[r, bend left, "\mathrm{Spec}"] \arrow[r, phantom, "\cong"] \arrow[d, "\text{if-then-else programs}"] & \mathrm{Stone\,Sp} \arrow[l, bend left, "\mathrm{Clop}"] \arrow[d, "\text{Identity Morphisms Only}"'] \\
\mathrm{Mnd}_{fin}(\mathrm{Set})^{op} \arrow[r, bend left, "\mathrm{Res} = \mathbb{B}(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Top\,Cof} \arrow[l, bend left, "\mathrm{Comp} = \Gamma(-)"]
\end{tikzcd}$$

The Terminal Comodel

The Observable behaviour of $w \in W$: $\langle - \rangle(w) : \int_{A \in Set} TA \to A$

Defn An (admissible) behaviour of T is a natural transformation

$$\beta : \int_{A \in Set} TA \to A \quad \text{s.t.} \quad \begin{aligned} &\beta(t \gg \text{pure } a) = a \\ &\beta(t \gg= u) = \beta(t \gg u(\beta(t))) \end{aligned}$$

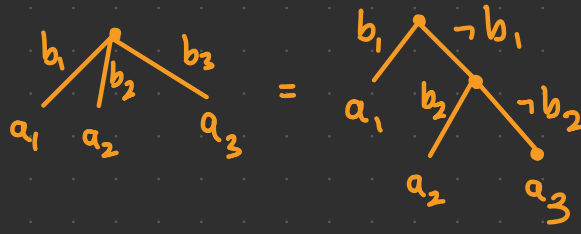


Prop The set $\mathbb{B}_0 T$ of behaviours is the terminal comodel, with $\langle t \rangle(\beta) = (\beta(t), \underbrace{\beta(t \gg -)}_{\partial_t \beta})$.

Examples 2^ω for T_{Bin}, S for $(S \times -)^S$, $\emptyset$ for $(- + E)$, $\mathcal{P}_{fin}$.

$$\beta(t) = \beta(\underbrace{b \gg b \dots \gg b}_{\exists n \in \omega}) \in 2$$

The Terminal Comodel of T_B

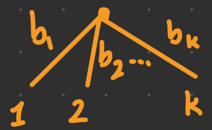


$\rightsquigarrow$

T_B generated by $T_B(2)$-elements,
in fact $B \cong T_B(2)$ with

$$b \longmapsto b(1,0)$$
$$\bot \sim \text{pure } 0$$
$$\top \sim \text{pure } 1$$
$$\neg b \sim b \gg= \lambda p.\, \text{pure}(1-p)$$
$$b_1 \wedge b_2 \sim b_1 \gg= \lambda p. \begin{cases} \bot & p=0 \\ b_2 & p=1 \end{cases}$$

$\beta\Big($



$\Big)$ = the unique i s.t. $\beta(b_i(1,0)) = 1$.

Each β determined by $\beta_2 : T_B 2 \to 2 \cong B \to 2 \cong \wp B$

Prop The admissible behaviours of T_B coincide with ultrafilters of B.

defn An ultrafilter $p \subseteq B$ is

- proper: $\bot \notin p$
- upwards-closed: $b_1 \in p$ & $b_1 \leq b_2 \Rightarrow b_2 \in p$
- meet-closed: $b_1 \in p$ & $b_2 \in p \Rightarrow b_1 \wedge b_2 \in p$
- ultra: $b \in p$ xor $\neg b \in p$.

$$\lhd t \rhd (p) = \big(p(t \gg -), p(t)\big) = \big(p, p(t)\big)$$

$\beta(\text{pure } 0) \neq 1$

$\beta(b_1) = 1 \Rightarrow \beta\left(b_1 \gg= \lambda p. \begin{cases} b_2 & p=0 \\ \top & p=1 \end{cases}\right) = \beta(b_1 \gg \top) = 1$

$\beta(b_1) = 1$ & $\beta(b_2) = 1 \Rightarrow \beta\left(b_1 \gg= \lambda p. \begin{cases} \bot & p=0 \\ b_2 & p=1 \end{cases}\right) = \beta(b_1 \gg b_2)$
$= \beta(b_2) = 1$

$\beta(\neg b) = \beta(b \gg= \lambda p.\, \text{pure}(1-p)) = 1 - \beta(b)$

The Terminal Topological Comodel

Defn The operational topology on $\mathbb{B}_\circ T$ is generated by sub-basic opens

$$[t \mapsto a] = \{ \beta \in \mathbb{B}_\circ T \mid \beta(t) = a \} \qquad \forall t \in TA,\ a \in A.$$

prop This makes $\mathbb{B}_\circ T$ the terminal topological comodel.

obs $[t \mapsto a]$ replaceable by $\left[t \gg\!= \lambda p. \begin{cases} \text{pure } 1 & p = a \\ \text{pure } 0 & p \neq a \end{cases} \mapsto 1 \right]$ so open sets generated by $[b \mapsto 1]$ $\forall b \in T2$.

prop The operational topology on $\mathbb{B}_\circ T_B$ coincides with the Stone topology on the set of ultrafilters $\mathrm{Spec}(B)$ of B.

examples $\mathbb{B}_\circ T_{Bin}$ is 2^ω with Cantor topology. $\mathbb{B}_\circ (S \times -)^S$ is discrete.

prop $\mathbb{B}_\circ T$ is a Stone space, for any finitary monad T.

i.e., Compact, Hausdorff, Clopen basis

$$t \in TA$$
$$\Downarrow$$
$$\exists t' \in T n.\ t = t'|_{n \to A}$$

The Behavior Category of a Monad

Clearly $\mathbb{B}_\circ T$ is "dual" to T. Can we make it into a structure which records the transitions without explicit reference to T?

Intuition Pick some $\beta \in \mathbb{B}_\circ T$.

Each $m \in TA$ induces a labelled transition $\beta \xrightarrow{m} \partial_m \beta$

s.t. $\beta \xrightarrow{t \gg= u} \partial_{t \gg= u} \beta = \beta \xrightarrow{t} \partial_t \beta \xrightarrow{u(\beta(t))} \partial_{u(\beta(t))} \partial_t \beta$.

For T_{Bin},

induces $1 :: 0 :: \beta \xrightarrow{b} 0 :: \beta \xrightarrow{b} \beta$

but

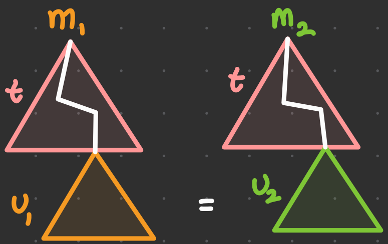
also induces $1 :: 0 :: \beta \xrightarrow{b} 0 :: \beta \xrightarrow{b} \beta$

So need to identify $1 :: 0 :: \beta \xrightarrow{m_1 \sim m_2} \beta$ by quotienting!

The Behavior Category of a Monad

Defn Let $\beta \in \mathbb{B}_0 T$. For $m_1, m_2 \in T1$, define

$$m_1 \overset{1}{\underset{\beta}{\sim}} m_2 \iff \exists\, t \in TA,\ u_1, u_2 : A \to T1. \left(m_1 = t \gg= u_1 \ \&\ m_2 = t \gg= u_2 \ \&\ u_1(\beta(t)) = u_2(\beta(t)) \right)$$



Let $\sim_\beta$ the least equivalence relation containing $\sim^1_\beta$.

Example

$T_{Bin} 1 / \sim_\beta \cong \mathbb{N}$ $\qquad$ $T_B 1 / \sim_\beta \cong 1$

$(S \times 1)^S / \sim_s \cong S$

Defn The behaviour category $\mathbb{B}T$ of T has

- A space of objects $\mathbb{B}_0 T$
- A space of morphisms $\mathbb{B}_1 T = \sum_{\beta \in \mathbb{B}_0 T} T1/\sim_\beta$

$$\rotatebox{90}{$\in$}$$
$$[m]_\beta$$

$id_\beta = [pure]_\beta$

$dom\ [m]_\beta = \beta$

$cod\ [m]_\beta = \partial_m \beta$

$[m]_\beta ; [n]_{\partial_m \beta} = [m \gg n]_\beta$

3 Computations as Global Sections

(of the source map dom: $\mathbb{B}_1 T \longrightarrow \mathbb{B}_0 T$)

$$\begin{tikzcd}
\mathrm{BoolAlg}^{op} \arrow[r, bend left, "\mathrm{Spec}"] \arrow[r, phantom, "\cong"] \arrow[d, "\text{if-then-else programs}"] & \mathrm{StoneSp} \arrow[l, bend left, "\mathrm{Clop}"] \arrow[d, "\text{Identity Morphisms Only}"] \\
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\end{tikzcd}$$

The topology on $\mathbb{B}_1 T$

Each $t \in TA$ manifests as a map

$$\begin{tikzcd}
\mathbb{B}_0 T \arrow[r, "\tilde{t}"] & \mathbb{B}_1 T \times A \\
\beta \arrow[r, mapsto] & \left([t \gg \mathrm{pure}]_\beta, \beta(t) \right)
\end{tikzcd}$$

which corresponds to an A-family of partial maps

$$\mathbb{B}_0 T \supseteq [t \mapsto a] \xrightarrow{\tilde{t}_a} \mathbb{B}_1 T$$

Requirement Each such $\tilde{t}_a$ be continuous, and as little else.

Defn The operational topology on $\mathbb{B}_1 T$ is generated by

$$[m \mid t \mapsto a] = \{ [m]_\beta \mid \beta(t) = a \}$$

prop $(\mathbb{B}_0 T, \mathbb{B}_1 T)$ is a topological category and each $\tilde{t}$ is continuous, and dom is a local homeomorphism.

The Monad of Global Sections

Let $\mathbb{C} = (\mathbb{C}_0, \mathbb{C}_1)$ be topological category.

Defn let

$$\Gamma\mathbb{C}(A) = \left\{ s : \mathbb{C}_0 \to \mathbb{C}_1 \times A \mid \forall c \in \mathbb{C}_0, \mathrm{dom}(\pi_{\mathbb{C}_1}(s(c))) = c \right\}.$$

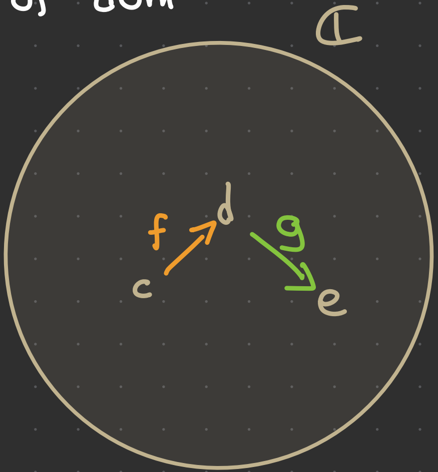
$$\Updownarrow$$

$\pi_{\mathbb{C}_1} \circ s$ section of dom

Prop $\Gamma\mathbb{C}$ is a monad on Set with

$$\mathrm{pure}\ a = \lambda c \in \mathbb{C}_0 . (id_c , a)$$

$$\begin{aligned} t \gg\!= u = \lambda c \in \mathbb{C}_0 .\ & \mathrm{let}\ (f : c \to d, a) = t(c)\ \mathrm{in} \\ & \mathrm{let}\ (g : d \to e, b) = u(a)(d)\ \mathrm{in} \\ & (f ; g , b) \end{aligned}$$



Moreover,

$$\natural : T \longrightarrow \Gamma\mathbb{B}T : t \longmapsto \left(\tilde{t} : \mathbb{B}_0 T \longrightarrow \mathbb{B}_1 T \times A\right)$$

Finite Lookahead

For $(S\times -)^S$,

$$\Gamma\mathbb{B}(S\times -)^S(A) = \{ s : S \longrightarrow \cancel{S}\times S\times A \mid \pi_0 s = id_S \} \cong (S\times A)^S$$

For T_B, $\Gamma\mathbb{B}T_B(A) = \{ s : Spec \longrightarrow \cancel{Spec\times 1}\times A \mid \pi_0 s = id_{spec} \} \cong A^{spec}$

$\cong T_B(A)$

(By compactness)

But for T_{Bin}, consider $s \in \Gamma\mathbb{B}T_{Bin}(A)$:

$$s(\beta) = (n, a) \iff s(\beta) \in [b^n \mid pure\ * \mapsto *] \times \{a\}$$

(by continuity) $\iff \exists b$ prefix β. $[b] \subseteq s^{-1}[b^n \mid pure\ * \mapsto *] \times \{a\}$

So by compactness, s is described by a finite dictionary

$\langle b_1 \mapsto (n_1, a_1), \dots, b_k \mapsto (n_k, a_k) \rangle$ s.t. $\{[b_i]\}_{i\le k}$ partition $\mathbb{B}_0T$

Finite Lookahead

s is described by a finite dictionary

$$\langle b_1 \mapsto (n_1, a_1), \dots, b_k \mapsto (n_k, a_k) \rangle \text{ s.t. } \{[b_i]\}_{i \leq k} \text{ Partition } \mathbb{B}_0 T$$

$\sim$

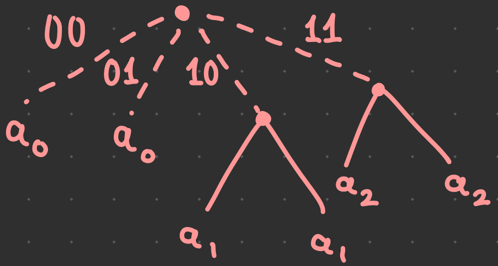
$0 \mapsto (1, a_1)$
$10 \mapsto (2, a_2)$
$11 \mapsto (3, a_3)$

Obs for $s = \tilde{t}$ of some $t \in T_{Bin}$,
$n_i \geq \mathrm{len}(b_i)$.

So there are other sections failing this condition!

$0 \mapsto (0, a_0)$
$10 \mapsto (1, a_1)$
$11 \mapsto (1, a_2)$

$\sim$



prop For T finitary, $s \in \Gamma \mathbb{B}T(A)$ factors as $s = t \gg= u$ where t only performs identity maps, and each $u(p) = \tilde{u}_p$ for some $u_p \in TA$.

Functoriality

Defn A morphism of Monads $S \xrightarrow{h} T$ is $h : \int_{A \in Set} SA \to TA$

such that

$$h(s_1 \gg\!=_S s_2) = h(s_1) \gg\!=_T \lambda a.\, h(s_2(a))$$

$$h(\text{pure}_S\ a) = \text{pure}_T\ a$$

i.e. an "interpreter".

induces

$$\begin{tikzcd}
\partial_{h(m)}\beta \arrow[r, mapsto] & \partial_m(\beta \circ h) \\
\beta \arrow[u, "{[h(m)]_\beta}"] \arrow[r, mapsto] \arrow[ur, phantom, "\leftsquigarrow"] & \beta \circ h \arrow[u, "{[m]_{\beta \circ h}}"'] \\
\mathbb{B}T \arrow[r, "\mathbb{B}h"] & \mathbb{B}S
\end{tikzcd}$$

Defn A cofuncter $\mathbb{C} \overset{F}{\rightsquigarrow} \mathbb{D}$ consists of

$F_0 : \mathbb{C}_0 \longrightarrow \mathbb{D}_0$ $\qquad$ $F_1 : \mathbb{C}_1 \longleftarrow \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{C}_0$

s.t.

$$\begin{tikzcd}
c \arrow[r, mapsto] & F_0(c) \\
c \arrow[u, "id"] \arrow[r, mapsto] \arrow[ur, phantom, "\overset{F_1}{\leftsquigarrow}"] & F_0(c) \arrow[u, "id"']
\end{tikzcd}
\qquad
\begin{tikzcd}
c_2 \arrow[r, mapsto] & d_2 \\
c \arrow[u, "F_1(g \circ f)"] \arrow[r, mapsto] \arrow[ur, phantom, "\leftsquigarrow"] & F_0(c) \arrow[u, "g \circ f"']
\end{tikzcd}
=
\begin{tikzcd}
c_2 \arrow[r, mapsto] & d_2 \\
c_1 \arrow[u, "F_1(g)"] \arrow[r, mapsto] \arrow[ur, phantom, "\leftsquigarrow"] & d_1 \arrow[u, "g"'] \\
c \arrow[u, "F_1(f)"] \arrow[r, mapsto] \arrow[ur, phantom, "\leftsquigarrow"] & F_0(c) \arrow[u, "f"']
\end{tikzcd}$$

i.e. a "simulation".

Conclusion

$$\begin{tikzcd}
\mathrm{BoolAlg}^{op} \arrow[r, bend left, "\mathrm{Spec}"] \arrow[r, phantom, "\cong"] \arrow[d, "\text{if-then-else programs}"] & \mathrm{StoneSp} \arrow[l, bend left, "\mathrm{Clop}"] \arrow[d, "\text{Identity Morphisms Only}"'] \\
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\end{tikzcd}$$

4 The Relationship with Ring Spectra (According to Diers)

Spectrum of a Commutative Ring

The Zariski-Grothendieck Spectrum of a Comm. Ring R is a local homeomorphism $\sum_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}} \longrightarrow \mathrm{Spec}(R)$

Universal property of $R_{\mathfrak{p}}$:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & L \\
R_{\mathfrak{p}} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

$$\sum_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{Hom}_{\mathrm{LocRing}}(R_{\mathfrak{p}}, L) \cong \mathrm{Hom}_{\mathrm{Ring}}(R, UL)$$

"measures the failure of U to have a left adjoint"

Diers: Spectra arise as indexing objects of multi-adjunctions.

A multi-adjunction for behaviours

Claim $\mathbb{B}_0 T$ arises from a left multi-adjoint to $U : \mathrm{Comod}_T \hookrightarrow \mathrm{RMod}_T$

Defn $\mathrm{RMod}_T = [\mathrm{Kl}(T), \mathrm{Set}]$

Example T itself manifests as

$$\begin{tikzcd} A \arrow[dd, "u"'] & & TA \arrow[dd, "(-)\gg= u"] \\ & \longmapsto & \\ TB & & TB \end{tikzcd}$$

Example Comodels $(W, \langle - \rangle)$ are exactly coproduct preserving right modules

$$\begin{tikzcd} A \arrow[dd, "u"'] & & W \times A \arrow[dd] & (w, a) \arrow[dd, mapsto] \\ & \longmapsto & & \\ TB & & W \times B & \langle u(a) \rangle (w) \end{tikzcd}$$

We can easily sus out the spectrum. Suppose

$$\sum_{p \in \mathrm{Spec}(M)} \mathrm{Hom}_{\mathrm{Comod}}(F_p M, W) \cong \mathrm{Hom}_{\mathrm{RMod}}(M, UW)$$

Then:

$$\mathrm{Hom}(M, U \mathbb{B}_0 T) \cong \sum_{p \in \mathrm{Spec}(M)} \mathrm{Hom}(F_p M, \mathbb{B}_0 T) \cong \mathrm{Spec}(M)$$

A multi-adjunction for behaviours

Claim $\mathbb{B}_\circ T$ arises from a left multi-adjoint to $U : \mathrm{Comod}_T \hookrightarrow \mathrm{RMod}_T$

Defn $\mathrm{R\,Mod}_T = [\mathrm{Kl}(T), \mathrm{Set}]$

Example Comodels $(W, \langle - \rangle)$ are exactly coproduct preserving right modules

$$\begin{tikzcd} A \arrow[d, "u"'] & \longmapsto & W \times A \arrow[d] & (w, a) \arrow[d, mapsto] \\ TB & & W \times B & \langle u(a) \rangle (w) \end{tikzcd}$$

Example T itself manifests as

$$\begin{tikzcd} A \arrow[d, "u"'] & \longmapsto & TA \arrow[d, "(-) \gg= u"] \\ TB & & TB \end{tikzcd}$$

We can easily sus out the spectrum. Suppose

$$\sum_{p \in \mathrm{Spec}(M)} \mathrm{Hom}_{\mathrm{Comod}}(F_p M, W) \cong \mathrm{Hom}_{\mathrm{RMod}}(M, U W)$$

Then: $\quad \mathrm{Hom}(M, U \mathbb{B}_\circ T) =: \mathrm{Spec}(M)$

Further: $\quad \mathrm{Spec}(T) \cong \mathrm{Hom}(T, U\mathbb{B}_\circ T) \overset{(\text{Yoneda})}{\cong} \mathrm{Hom}(\text{よ} 1, U\mathbb{B}_\circ T) \cong U\mathbb{B}_\circ T(1) \cong \mathbb{B}_\circ T$

Dependence on T rather unsatisfying... ☹

The analogy with Rings

$\mathrm{CRing} \sim \mathrm{RMod}_T$

$\mathrm{LocRing} \sim \mathrm{Comod}_T$

$\mathrm{Spec}(R) \sim \mathbb{B}_0 M := \mathrm{Hom}_{\mathrm{RMod}}(M, U\mathbb{B}_0 T)$

$\sum_{\mathfrak{p}\in \mathrm{Spec}(R)} R_{\mathfrak{p}} \xrightarrow{lh} \mathrm{Spec}(R) \sim \mathbb{B}_1 M := \sum_{\mathfrak{p}\in \mathbb{B}_0 M} M1/{\sim_{\mathfrak{p}}}$

$$\begin{tikzcd}
 & \mathbb{B}_1 M \arrow[dl, "lh"'] \arrow[dr] & \\
\mathbb{B}_0 M & & \mathbb{B}_0 T
\end{tikzcd}$$

with $\mathbb{B}T$-action

$\triangleright : \mathbb{B}_1 M \times_{\mathbb{B}_0 T} \mathbb{B}_1 T \to \mathbb{B}_1 M$

Further Work

- Localic version of $\mathbb{B}T$
  - ↳ For Cole-related reasons, need to define inside arbitray Gr Topos.
  - ↳ Works better for infinitary T (non-spatial locales)
- $\Gamma \mathbb{B} T$ adds read-only operations. This seems to be artefact of demanding a "state space" $\mathbb{B}_0 T$.
  - ↳ Suggests replacement of top. spaces/locales by quantales whose "opens" are not read-only.

$\mathbb{B}_1 T$ is a Quantale with $\mathcal{U} * \mathcal{V} = \left\{ gf \;\middle|\; \begin{matrix} f \in \mathcal{U},\ g \in \mathcal{V} \\ \operatorname{cod}(f) = \operatorname{dom}(g) \end{matrix} \right\}$

↳ Based on work by P. Resende on Top. Groupoids & Quantales.

What can we do with this?

A The relationship with Ring Spectra

(According to Cole)

or:

How I learned to stop worrying and love Topos Theory

The universal property of the operational topology

Diers' approach explains the underlying set of Spec(R) and Spec(M), but not the topology. Cole explains both:

Warning: Cole's 1-cells go in the opposite direction!

$$\begin{tikzcd}
\mathrm{LocGrTop} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{CRingGrTop} \arrow[l, bend right, "F"']
\end{tikzcd}$$

$$\rotatebox{90}{$\in$}$$

$$(\mathcal{E}_1, R_1) \xrightarrow{(f,h)} (\mathcal{E}_2, R_2)$$

$f : \mathcal{E}_2 \to \mathcal{E}_1$ & $h : f^* R_1 \to R_2$.

In particular: $F(\mathrm{Set}, R) \simeq (\mathrm{Sh}(\mathrm{Spec}(R)), \mathcal{O}_R)$

Can we do this for Comodels & Right Modules?

The universal property of the operational topology

The bicategories $\mathrm{RMod}_T(\mathrm{Gr\,Top})$ and $\mathrm{Comod}_T(\mathrm{Gr\,Top})$:

$$\begin{tikzcd}
\mathcal{E} \\
\mathrm{Kl}(T) \arrow[u, "M"]
\end{tikzcd}$$

Objects

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "f^*"] & & \mathcal{F} \\
& \mathrm{Kl}(T) \arrow[ul, "M"] \arrow[ur, "N"'] \arrow[u, phantom, "\Rightarrow"] &
\end{tikzcd}$$

1-cells

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left=50, "g^*"] \arrow[rr, "f^*"'] \arrow[rr, phantom, bend left=25, "\Downarrow"] & & \mathcal{F} \\
& \mathrm{Kl}(T) \arrow[ul, "M"] \arrow[ur, "N"'] \arrow[u, phantom, "\Rightarrow"] &
\end{tikzcd}$$

2-cells

inherit posetal 2-cells

Prop $\mathrm{Fib}\,\mathbb{B}T\,\mathrm{Sp} \simeq \mathrm{Comod}_T(\mathrm{Sp\,Top})$

proof Essentially by $\mathrm{Sh}(X) \simeq \mathrm{LH}/_X$.

The universal property of the operational topology

$$\begin{tikzcd}\mathrm{Consd}_T(\mathrm{Gr\,Top}) \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{RMod}_T(\mathrm{Gr\,Top}) \arrow[l, bend right, "F"']\end{tikzcd} \; ?$$

Strategy: let $F(\mathcal{E}, M) = (\mathcal{F}, \mathcal{O}_M)$

- Consider the case $\mathcal{E} = \mathrm{Set}$, but work constructively

✓

- Construct localic version of $\mathbb{B}M$

Bonus: get generalization to infinitary monads, which is NOT spatial.

- By previous prop, get $\mathcal{O}_M$.
- Repeat the above, but working internally in $\mathcal{E}$
- ???
- <u>PROFIT</u>: Get $\mathcal{F}$ as "$\mathcal{E}$-internal sheaves on $\mathbb{B}_0 M$"

  and $\mathcal{O}_M$ as "the internal sheaf corresponding to $\mathbb{B}M$"

References

- John Power, Olha Shkaravska 2004
  "From Comodels to Coalgebras: State and Arrays"
- Richard Garner 2022
  "The costructure - cosemantics adjunction for comodels for comp. effects"
- Richard Garner 2021
  "Stream processors & comodels"
- Yves Diers 1981
  "Some spectra relative to functors"
- Julian Cole Republished 2016
  "The bicategory of topoi and spectra"