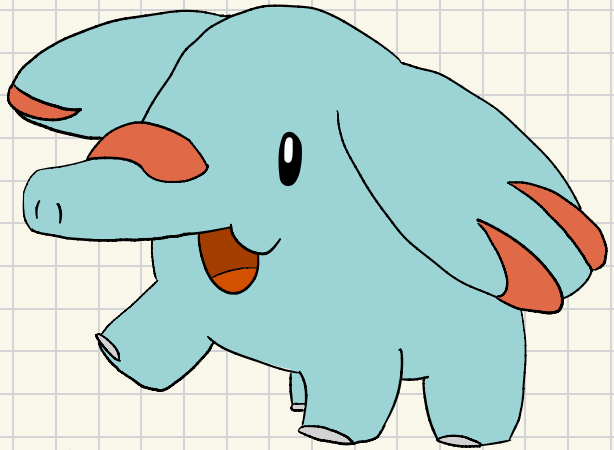
The Stone Spectrum of a Monad

Richard Garner
Alyssa Renata (me!)
Nicolas Wu



29 March 2026 PSSL 112 Nottingham

Part 1: Motivation

Monads encode Effectful Operations

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code.txt

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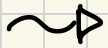

algebraic operations

$put_{x,15}($

)

unary operation

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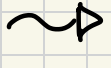

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  $get_x(\lambda m.$
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N-ary operation

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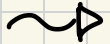

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  $get_x(\lambda m.$
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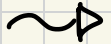

algebraic operations

$$put_{x,15}(\ get_x(\lambda m.\ put_{y,m+1}(\ get_y(\lambda n.\ \begin{cases} \text{true} & n = 17 \\ \text{false} & n \neq 17 \end{cases})\ )\ )\ )$$

"true" and "false" are variables in the algebraic sense

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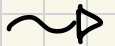

algebraic operations

$$
\begin{aligned}
&\mathrm{put}_{x,15}(\\
&\quad \mathrm{get}_x(\lambda m.\\
&\qquad \mathrm{put}_{y,m+1}(\\
&\qquad\quad \mathrm{get}_y\left(\lambda n. \begin{cases} \text{true} & n = 17 \\ \text{false} & n \neq 17 \end{cases}\right)\\
&\qquad)\\
&\quad)\\
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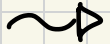

algebraic operations

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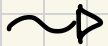

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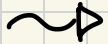

algebraic operations

$put_{x,15}($

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$get_y(n.$ {true n = 17 {false n ≠ 17 } 16 }

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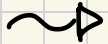

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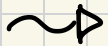

algebraic operations + equations

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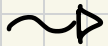

algebraic operations + equations =: $\mathbb{T}$ alg. theory

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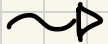

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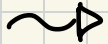

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$T\{true, false\}$

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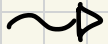

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- If Y = output values, TY = computations returning values in Y
- Kleisli map $X \to TY$ = program $X \to Y$

What is being effected?

Let $\mathcal{E}$ be category with <u>copowers</u> $\coprod_X E$ for all $X \in \mathrm{Set}$.

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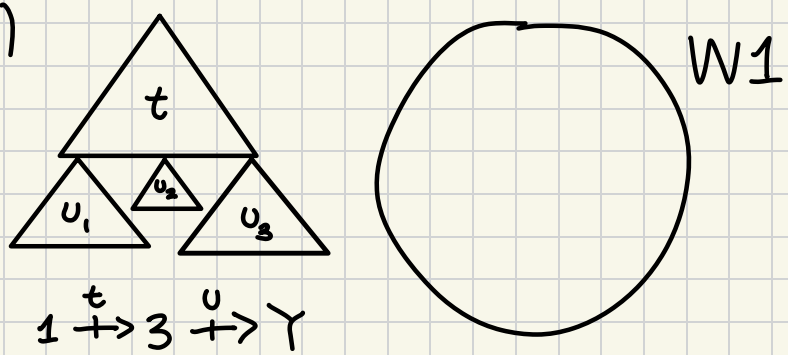
Example ($\mathcal{E} = \mathrm{Set}$)



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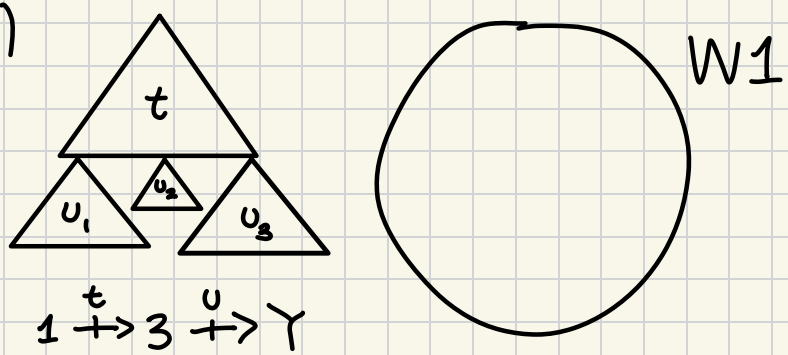

Comodels are Transition Systems!

Classification of Comodels

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[Garner '21]

$$\mathrm{Comod}_T(\mathrm{Set}) := [\mathrm{Kl}(T), \mathrm{Set}]_{\perp\!\!\!\perp}$$

$$\simeq [\mathbb{B}T, \mathrm{Set}]$$ Presheaf Cat!

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Observation B Bool. Alg. $\rightsquigarrow$ T_B Monad of B-partitions

B-tests / if $b \in B$ then... else...

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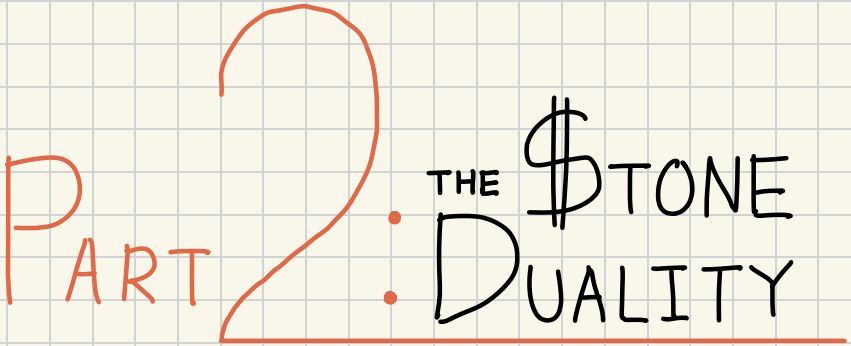
PART 2: THE $TONE DUALITY

The Topological Behaviour Category

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A: Classify $\mathrm{Comod}_T(\mathrm{Top})$!

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suffice to put correct topology on $\mathbb{B}_0\mathbb{T}$ and $\mathbb{B}_1\mathbb{T}$ ⇝ Internal Category

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Theorem

$$\mathrm{Mnd}_{\omega}(\mathrm{Set}) \simeq \mathrm{TopCat}^{op}$$

with $\mathbb{B}$ and Γ_{ω}

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$S \xrightarrow{\varphi} T$

$\mathbb{B}\varphi$

$\mathbb{B}T \rightsquigarrow \mathbb{B}S$

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$S \xrightarrow{\varphi} T$ $\quad (\Gamma_{\omega}\mathbb{C})n = \{$ global sections (to source map) $\mathbb{C}_0 \to \coprod_n \mathbb{C}_1 \}$

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$\mathbb{C}$

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Remark

Indeed,

$\mathbb{B}_0\mathbb{T}_B = \mathbb{B}_1\mathbb{T}_B \cong \mathrm{Spec}(B)$

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$\mathbb{B}_0T_B = \mathbb{B}_1T_B \cong \mathrm{Spec}(B)$

In fact,

$\mathbb{B}_0T$ is always Stone!

(for T finitary)

The Localic Behaviour Category

- We can redo this for $\mathrm{Comod}_T(\mathrm{Loc})$, to get

$$\mathrm{Mnd}_{\mathrm{acc}}(\mathrm{Set}) \underset{\mathbb{I}}{\overset{\mathbb{B}}{\rightleftarrows}} \mathrm{Loc}\mathrm{Retro}^{\mathrm{op}}$$

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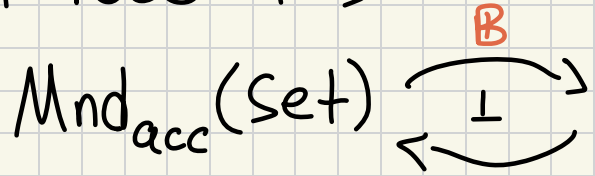

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- This clarified $\mathbb{B}_1 T \xrightarrow{\mathrm{src}} \mathbb{B}_0 T$ as a sheaf generated by $T1$, which let us prove:

theorem $\mathrm{CCMnd}_{acc}(\mathrm{Set}) \simeq \mathrm{AmpleLocRetro}^{op}$

- Strongly zero-dim AKA "superStone"
- local homeomorphism src

[uses a lot of technology from Garner '24 + '25]

If interested, Check out our preprint!

"Stone Duality for Monads"

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(If not, check it out anyway? 🥺)

PART 3: TOWARDS ZARISKI & GROTHENDIECK

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Leave this for future work. For now, let's compare

ZSpec(R) vs BT

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$$\begin{array}{ccc} (R, \mathrm{Set}) & \xrightarrow{(h,\, F)} & (L, \mathcal{E}) \\ (\mathcal{O}, \Gamma)\big\downarrow & & \\ (\mathcal{O}R, \mathrm{Sh}(\mathbb{Z}\mathrm{spec}(R))) & & \end{array}$$

i.e. $F : \mathcal{E} \leftarrow \mathrm{Set}$, $h : F^{*}R \to L$

in Ringed Gr Top

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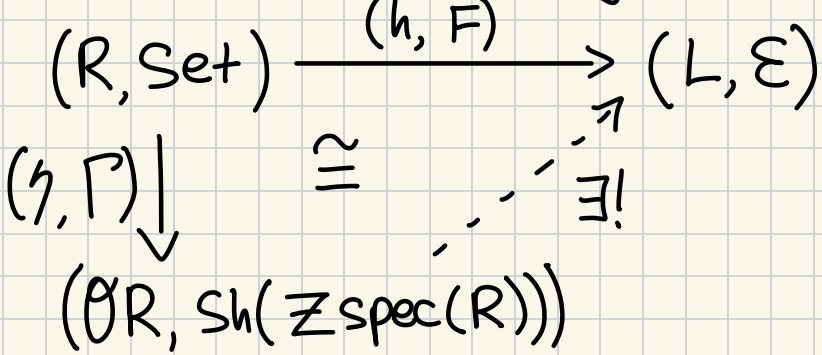

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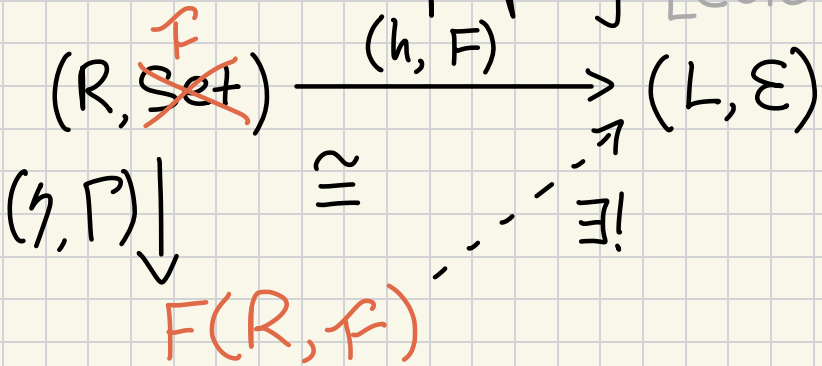

i.e. $F : \mathcal{E} \leftarrow \mathrm{Set}$, $h : F^{*}R \to L$ in Ringed Gr Top

Theorem

$$\mathrm{Ringed\,Gr\,Top} \underset{\longleftarrow}{\overset{F}{\longrightarrow}} \mathrm{Loc\,Ringed\,Gr\,Top} \quad (\dashv)$$

The free comodel lives over B.T

The free comodel lives over $\mathbb{B}_\bullet T$

Remark

Comodels are special cases of T-actions $M \circ T \to M$,
Since $\mathrm{Act}_T(\mathcal{E}) \cong [Kl(T), \mathcal{E}] \hookleftarrow [Kl(T), \mathcal{E}]_{LL} =: \mathrm{Comod}_T(\mathcal{E})$.

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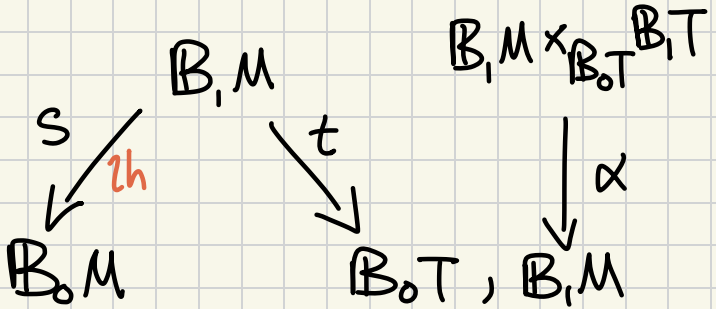
The construction $\mathbb{B}$ can be generalized to $\mathbb{B}M$, a fibered $\mathbb{B}T$-space:



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$$\text{fibered } \mathbb{B}T\text{-space} := \left\{ \left(\begin{array}{ccccc} & & E & & E \times_{\mathbb{B}_0 T} \mathbb{B}_1 T \\ & S \swarrow & & t \searrow & \downarrow \alpha \\ B & & & & \mathbb{B}_0 T, \quad E \end{array} \right) \right\}$$

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$\simeq$ T-Comodelled LocTop

$\subseteq$ T-Comodelled GrTop

$$\| \quad \{(\mathcal{E}, W : Kl(T) \to \mathcal{E})\}$$

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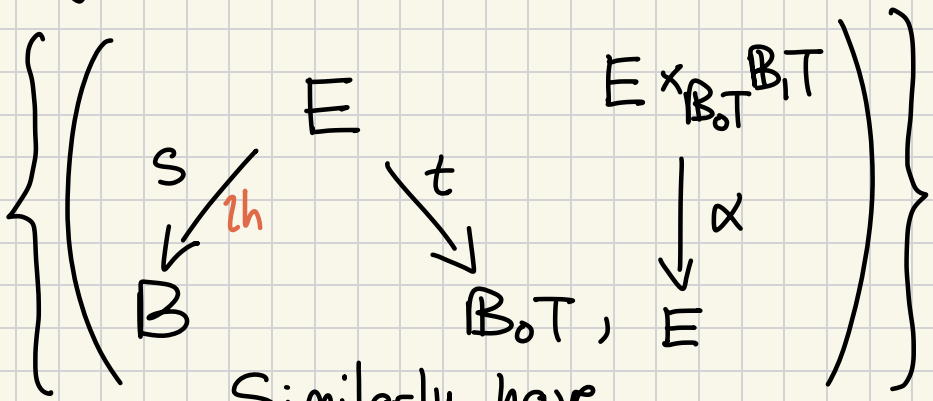


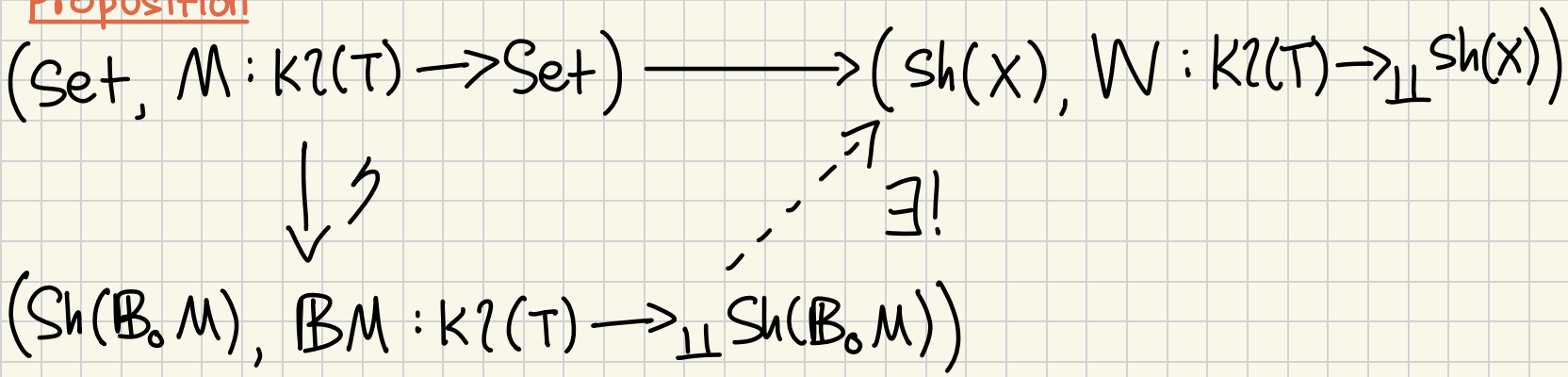
$\simeq$ T-Comodelled LocTop
$\subseteq$ T-Comodelled GrTop
$=$ $\{(\mathcal{E}, W : K\ell(T) \to \mathcal{E})\}$ $\hookrightarrow$ T-Actioned GrTop

Similarly, have

The free comodel lives over $B_0 M$

The T-comodel BM admits a universal property

Proposition



The free comodel lives over $\mathbb{B}_0 T$ M

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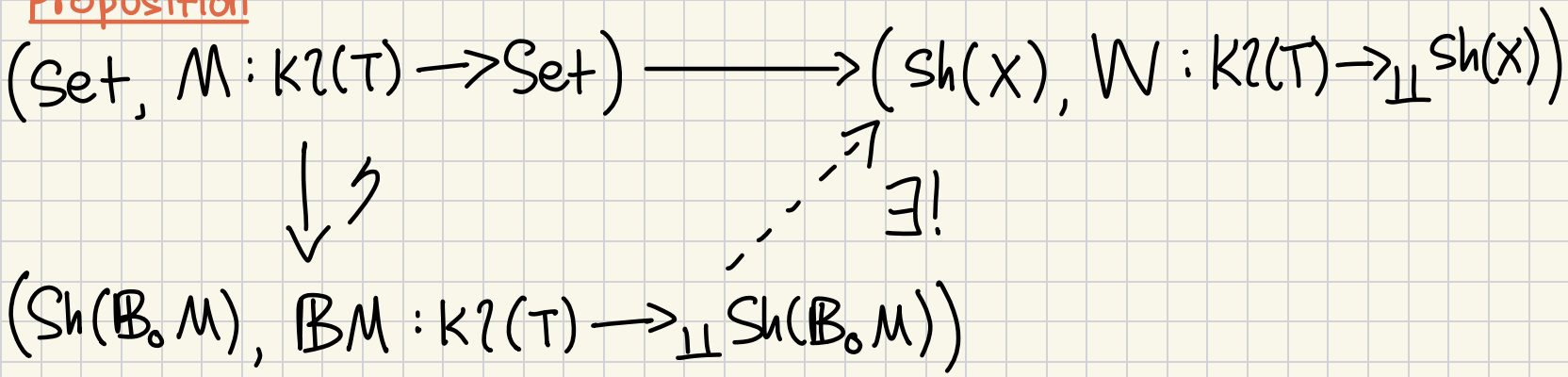

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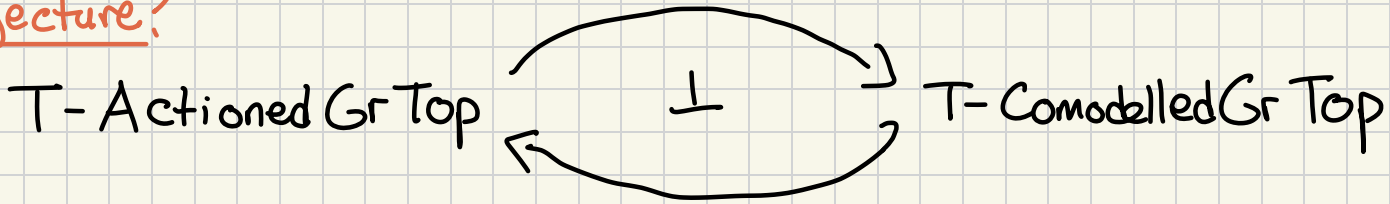

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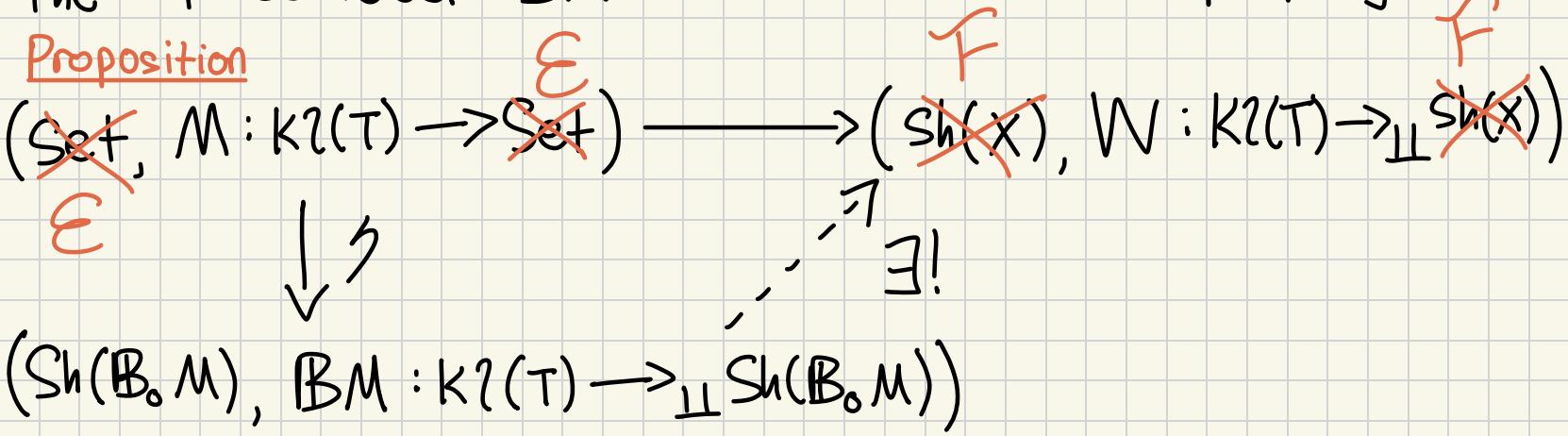

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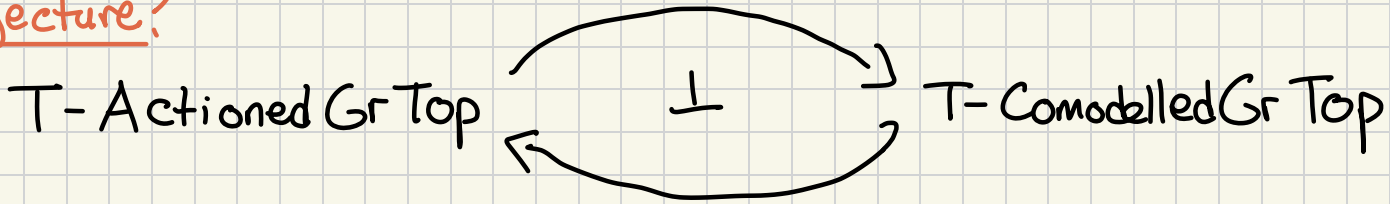

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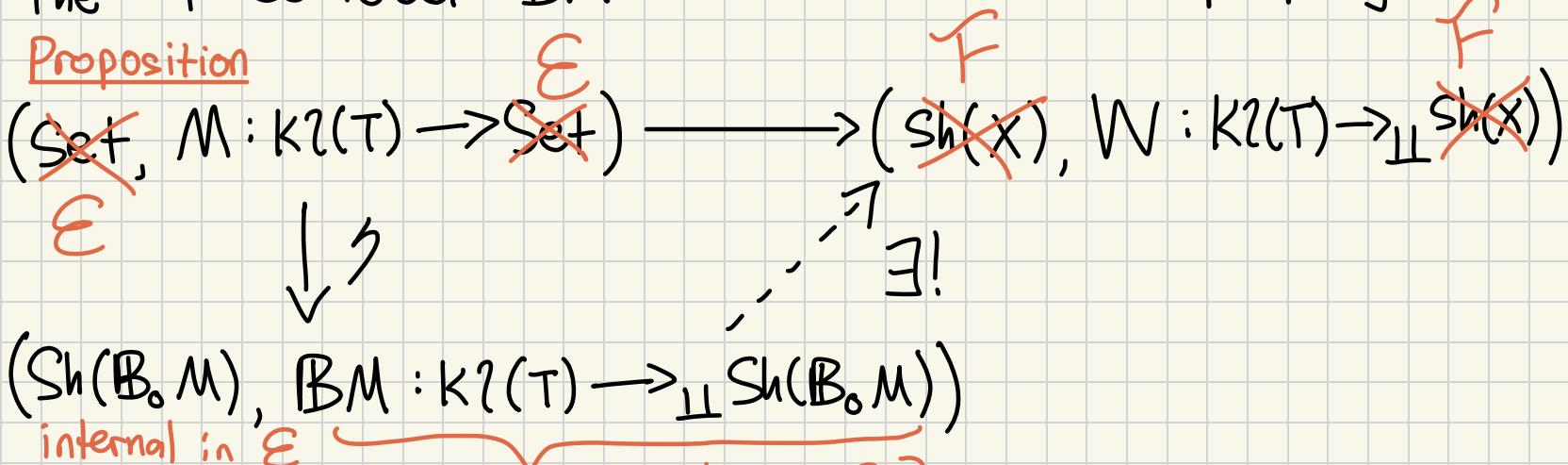

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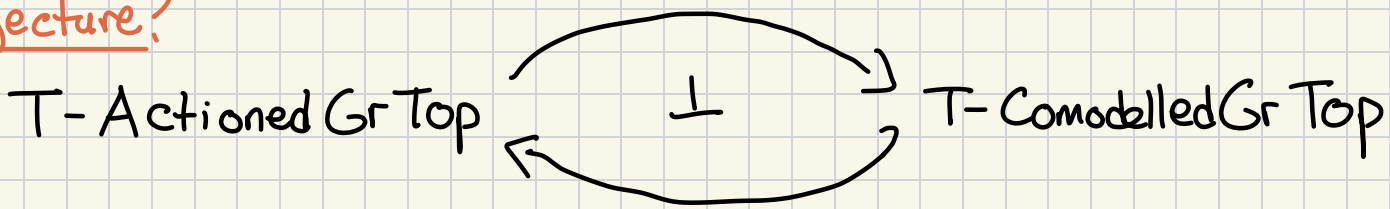

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