Homotopy Theory of Computable Spaces

MSc Logic Thesis Defense

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- By contrast, HoTT usually has:

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- Notions of computable set give more interesting interpretation.
- Question: Is univalence consistent with impredicativity? What would its models look like?
- Answer(?): Some notion of computable space.

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Let Equ be category of equilogical spaces with pointwise equivalence classes of equivariant maps.

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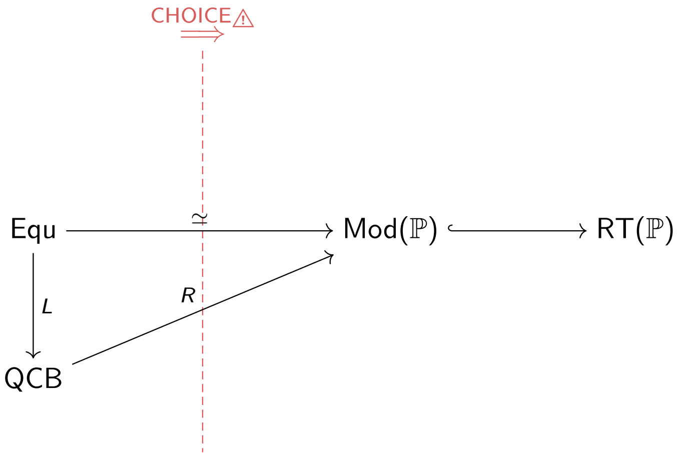
caveat: R needs choice of pseudobase enum. for each $X \in$ QCB

Some Context: Realizability



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Theorem (Strøm 1972)

The category of topological spaces has a model structure where W *is the class of homotopy equivalences.*

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A similar results hold for QCB:

Theorem (Chapter 5)

The category of QCB spaces has a model structure where W *is the class of homotopy equivalences.*

Ideally: there should be a homotopy theory on Equ such that both model structures interact well over $L \dashv R$. At least, L should send equivalences in Equ to homotopy equivalences in QCB.

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Theorem (Theorem 6.23)

There is no model structure on Equ *where* W *is the class of maps $f : X \to Y$ for which there is a $g : Y \to X$ such that $gf \simeq^*_{\mathbf{I}} id$ and $fg \simeq^*_{\mathbf{I}} id$.*

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There is a path category structure on Eql *whose homotopy category is* Equ. *Its path object is* $X^{\bullet\sim\bullet} := \{ (x, x') \mid x \sim x' \}$.

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WARNING: multiple notions of path at play. Let $\mathcal{I} = \bullet \sim \bullet$.

The Homotopy Realizability Topos

Inspired by Benno showing $\mathsf{RT}(\mathbb{P})$ is homotopy category of another category $\mathbb{RT}(\mathbb{P})$,

Theorem (Theorems 6.17 & 6.18)

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In fact, $\mathsf{Eql} \hookrightarrow \mathbb{RT}(\mathbb{P})$ *preserves* <u>*and reflects*</u> *all path category structure.*

<u>CAVEAT</u>: $\mathsf{Eql} \hookrightarrow \mathbb{RT}(\mathbb{P})$ depends on choice of basis enumeration...

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- Problem: I cannot find any research on such a notion.

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- Can find homotopy theories on equilogical spaces, but not quite one incorporating both $\mathcal{I}$ and $\mathbf{I}$.
- Need to develop the homotopical perspective on realizability.
- Also want a general theory of amalgamation in homotopy theory. Experience suggests paying attention to higher homotopies, i.e. higher categorical structure.

Other Stuff

- Experiment: let's put a path category structure on Eql where each $X^{\mathbf{I}^n}$ is a path object.
- Have to enforce many interaction conditions between $\mathbf{I}$-paths and $\mathcal{I}$-paths.
- Result: restrict to "boring" equilogical spaces.

- With deeper understanding of $\mathrm{RT}(\mathbb{P})$ and $\mathbb{RT}(\mathbb{P})$, see what when wrong in defining MY, following de Jong & van Oosten.
- From realizability perspective, underlying set of MY should be not-necessarily-continuous functions $\mathbf{I}^n \to Y$.
- The realizers of each such function is witnesses to its continuity.
- Moral: in $\mathrm{RT}(\mathbb{P})$ "computable" means continuous.
- Not strictly necessary, but homotopical perspective on realizability would greatly help.