Homotopy Theory of Computable Spaces

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis, we develop the homotopy theory of equilogical spaces and QCB (**Q**uotients of **C**ountably-**B**ased) spaces. An equilogical space is a T_0 countably-based space equipped with an equivalence relation, while a QCB space is the quotient of some equilogical space by its relation. We show that from any QCB space X we can reconstruct an equilogical space inducing X under quotienting, thus exhibiting the QCB spaces as a reflective subcategory of the equilogical spaces.

We construct a Quillen model structure for QCB spaces in which the weak equivalences are homotopy equivalences. We then seek a corresponding homotopy theory for equilogical spaces, but the notion of homotopy for equilogical spaces induced by the unit interval $[0, 1]$ is *not* transitive. Hence, we study instead a notion of homotopy corresponding to taking transitive closure of $[0, 1]$-paths.

To accomodate this study, we prove that the category of equilogical spaces can be viewed as a homotopy category induced by a computational notion of homotopy. In fact, since the category of equilogical spaces embeds into a realizability topos, this result coincides with an existing result that realizability toposes are homotopy categories [Ber20]. From this point of view, we sketch a proof for obtaining a path object corresponding to the aforementioned transitive-closure homotopy.

Contents

Chapter 1

Introduction

In this thesis, we develop the homotopy theory of two notions of computable space, with the long-term goal of eventually using them to establish some models of homotopy type theory. In this introduction, we briefly motivate this goal, as well as outline the structure and contributions of the thesis.

Homotopy Type Theory

In the past few decades, there has been a push to converge the traditionally discrete study of logic and computation with the geometric/topological study of spaces. One of the most recent manifestations of this trend is homotopy type theory [The13], which refers to the interpretation of types in a formal type theory as spaces, as well as the addition of type-theoretic primitives justified by this interpretation. With these constructions, homotopy type theory serves as a foundation of mathematics taking spaces as primitive objects, instead of sets.

The Abridged History

Homotopy type theory was originally developed in the context of a type theory due to Martin-Löf (Martin-Löf Type Theory; MLTT) [Mar75; MS84]. In Martin-Löf's original conception, the types were to be interpreted as both sets, on which we can build the mathematical objects we are interested in, and also simultaneously as the propositions with which we state things about these mathematical objects. For example, conjunction and disjunction also simultatenously play the role of the Cartesian product and the disjoint sum of sets. Of particular interest is the identity type, logically playing the role of equality, and set-theoretically interpreting the (subsingleton) set of equalities between two elements.

This set-theoretic interpretation of the identity type is rather poor, but it was later observed that in MLTT, we can more generally[1] interpret types as spaces and identity types as the space of paths between two elements [AW09; HS98]. With this view, one can define various notions from algebraic topology, in particular the notion of homotopy equivalence between two types, seen as spaces. Two spaces are homotopy equivalent if there is a deformation from one to the other and vice versa. From these preliminary developments, homotopy type theory began.

Universes & Univalence

How might one seek to quantify and reason over all types? To do this, Martin-Löf introduced a *universe of types* $\mathcal{U}$, itself a type. The terms of type $\mathcal{U}$ are *names* for the actual types, so quantification over $\mathcal{U}$ is (indirectly) quantification over all types. Post Russell's paradox, we are primed to immediately ask: does $\mathcal{U}$ contain a name for itself, i.e. should $\mathcal{U}$ be a type of all types? Indeed, Martin-Löf initially included such a self-referential name in his type theory, but this was quickly shown to be inconsistent by a modification of Russell's paradox [Gir72]. Therefore, Martin-Löf modified his theory so that $\mathcal{U}$ does not name itself.

In the context of homotopy type theory, the identity type over some type X is the "space of paths" between two elements of that type. But what about the identity type over $X = \mathcal{U}$? It turns out that MLTT does not say very much on what this should be, so the notion is rather underdetermined. However, we already have a notion of identity between two types=spaces, namely homotopy equivalence. Therefore, the *univalence axiom* was added which (in its slogan form) determines the identity type on $\mathcal{U}$ as the space of homotopy equivalences between two types=spaces. Then we can view $\mathcal{U}$ as the large space of small spaces.

[1] viewing a set as a discrete space, there is a (unique) path between two elements iff the elements are equal.

The Calculus of Constructions

As explained in [Luo94], the inconsistency of $\mathcal{U}$ being the type of all types can be distentangled into a conflict between the following two ideas, which are:

1. Types as propositions - Martin-Lof's requirement that every type determines a proposition [Mar71, p. 2], not just vice versa.

2. Impredicativity - the requirement that we can quantify over the type of all propositions.

Together, they require for $\mathcal{U}$ to be a type of all types. While Martin-Löf found a way out by eliminating the second requirement, what theory do we get by eliminating the first requirement instead? That is to say, every proposition is considered a type, but *not vice versa*.

This type theory, called the calculus of constructions (CoC) [CH88], has a type Prop of all propositions but since not all types are propositions, Prop is not a type of all types. In particular, to prevent the paradox, Prop does not contain (a name of) itself. The ability to state a proposition quantifying over all propositions means that CoC is a higher-order logic, and from a practical perspective of formalising mathematics in type theory, this makes it more convenient to use. For example, the popular proof assistant Lean is based on CoC.

While philosophically Prop is intended to be the type of propositions and so one should not carry out mathematical constructions in Prop, in practice there is nothing stopping this. In fact, the impredicativity makes it a lucrative place to construct structures, using the so-called impredicative encodings of data types, e.g. [BB85]. In order to support this practice, we could try to impose the univalence axiom on Prop, so that we can develop a clear understanding of the identity between these structures constructed in Prop. But to do this, we must first ask whether univalence is consistent with impredicativity. More generally, is impredicativity compatible with a homotopy-theoretic interpretation of CoC? Such questions of consistency and compatibility can be answered by studying the models of CoC.

The Standard Model

The impredicativity of Prop turns out to be rather strongly incompatible with a set-theoretic interpretation [Rey84], at least of set theory formulated in classical logic. However, interpretations of Prop can be made in constructive/intuitionistic set theory [Pit87]. One way of obtaining such notions of constructive sets is to consider some variation on the informal notion of "computable set". Indeed, in [LM91] a model of CoC is constructed using *assemblies*, which are sets realized/implemented by computation in Turing machines. The propositions are modelled by an essentially small subclass of the assemblies called the *modest sets*, and Prop is modelled by an "assembly of modest sets". This can be regarded as the "standard" model of CoC.

Now, in order to model HoTT, we need spatial or more specifically homotopy-theoretic models. In order to model CoC, we need to equip the model with some flavour of computation. Hence, if we are seeking a homotopy-theoretic model of CoC (where ideally Prop is both univalent and impredicative), then we should substitute the "set" for "space" in "computable set". That is, look for some appropriate notion of *computable space*, and develop it into a homotopy-theoretic model of CoC. The first step towards this goal is of course to just develop the homotopy theory of computable spaces, which is the main objective of this thesis. We leave the heavy type-theoretic concerns for the future, but still focus on homotopy-theoretic ideas that are relevant for type theory.

Two Notions of Computable Spaces

The following line of reasoning leads us to consider some notions of computable spaces. Consider what it means for a topological space to be computationally representable. To represent the topology computationally, one needs to assign a natural number code to the open sets, or at least to generating open sets. So the computationally representable topologies are those which are generated by enumerably many generators, i.e. a countable basis. Furthermore, we would want a separation condition so that we do not have to explicitly keep track of individual points of the space, of which there can be uncountably many. With the T_0 condition, we can keep track of points as families of basic opens, encodable as infinite sequences/sets of numbers. Hence, as a first approximation, we can consider a computationally representable space to be a countably-based T_0 spaces, or ωT_0 for short.

One of the most important tools in topology is to take quotients of spaces, i.e. gluing of spaces. For example, up to homotopy equivalence any space can be replaced by a well-behaved space called a *CW complex*, obtained by gluing multiple copies of n-dimensional disks. However, in general taking quotients can break either the countable basis property or the T_0 property. In Appendix A, we characterize the CW complexes that are ωT_0 as those that are countable and for which each n-disk is only glued to finitely many other disks. This rules out a lot of CW complexes, so it seems ωT_0 spaces are not suitable for doing homotopy theory.

One way to fix the lack of quotients would be to "freely" add quotients, by considering objects $(X, \sim_X)$ where X is an ωT_0 space equipped with an equivalence relation $\sim_X$. We call such a pair an *equilogical space* [BBS04; Sco96]. The idea is that $(X, \sim_X)$ represents the quotient $X/_{\sim_X}$. Equilogical spaces are the first notion of computable space we will focus on in this thesis. The equilogical spaces originally arose in the context of *realizability* [Bau00], and in fact they are modest sets - not for Turing computability but for a model of computation called *Scott's Graph Model*.

So, the issue with quotienting is fixed, but an equilogical space isn't really a topological space anymore. However, given an equilogical space $(X, \sim_X)$, we can still ask what topological space we get by actually carrying out the quotient $X/_{\sim_X}$. We would like to maintain the T_0 separation condition, so we can further apply the T_0 quotient $(X/_{\sim_X})/_0$. The spaces that arise in this way are called the QCB (**Q**uotients of **C**ountably-**B**ased) spaces. QCB spaces are the second notion of computable space we will focus on in this thesis.

Conceptually, we can relate the two notions in the following way. We can view a QCB space as a computationally representable space, and an equilogical space inducing it as one of its computational representations. Because the QCB spaces are actually topological spaces, the classical notions of homotopy and homotopy equivalence can be defined for QCB spaces, providing us with a *classical* homotopy theory. Then, following our conception of equilogical spaces as computational representations, by a classical homotopy theory on equilogical spaces, we mean one that "represents" the classical homotopy theory of QCB spaces. That is, homotopies of equilogical spaces, however we define them, should induce homotopies in QCB.

Overview of this Thesis

Outline

- chapter 2 fixes some topological notions and imports some basic general topology theorems. It also briefly introduces domain theory, necessary for understanding and working with Scott's graph model.

- chapter 3 introduces the two *categories* of equilogical spaces and QCB spaces. We establish some basic results about limits, colimits and exponential objects. Using Scott's graph model, we are able to give an equivalent definition of QCB spaces in terms of intrinsic properties (i.e. without alluding to equilogical representations). In particular, we construct a fully faithful right adjoint to the quotienting functor from equilogical spaces to QCB spaces.

- In chapter 4, we begin our homotopical investigation by introducing and distinguishing model structures and path categories as two formal presentations of a "homotopy theory" on a category, along the way motivating basic definitions such as homotopy, fibrations and cofibrations. We also show a method of obtaining path categories from a sufficiently well-behaved interval object.

- In chapter 5, we construct a model structure on the category of QCB spaces based on the usual notion of homotopy using the standard unit interval $[0, 1]$. This model structure is based on Strøm's model structure for the category of topological spaces, although the proof does not immediately apply to QCB spaces because colimits and limits in the category of QCB spaces are computed differently than in the category of topological spaces.

- In chapter 6, we explore the classical homotopy theory of equilogical spaces. We immediately observe that while $[0, 1]$ can be seen as an equilogical space with a discrete relation, the notion of homotopy it induces in the category of equilogical spaces is not transitive. This leads to many difficulties, so a large part of the chapter is focused on introducing a homotopical perspective of realizability in order to frame the problem better. At the end of the chapter, we use this perspective to suggest a way forward in establishing a classical homotopy theory for equilogical spaces.

Contributions

The new results in this thesis are:

1. The adjunction between the category of equilogical and QCB spaces (Corollary 3.27).
2. The construction of a path category structure from a strict interval object with a contraction map (Theorem 4.24).
3. The model structure on the category of QCB spaces (chapter 5).
4. The presentation of the category of equilogical spaces Equ as a homotopy category of another category Eql (Corollary 6.7), and its embedding into the existing result that the realizability topos over Scott's graph model is a homotopy category (Theorem 6.17, Theorem 6.18).
5. The non-existence of model structures on the category of equilogical spaces Equ with length-global homotopy equivalence (Theorem 6.23).
6. The path category structure on mixed fibrant objects on Eql (Theorem 6.31).

Chapter 2

Topology, Domain Theory & Realizability

In this chapter, we give some basic general topological definitions in order to fix notation and disambiguate some topological terminology that may be defined in many different (non-equivalent) ways in the literature. We also prove some general lemmas which would otherwise distract from the exposition of the latter chapters. We therefore suggest that this chapter is skipped on first reading, and to return to it as necessary.

2.1 General Topology

Basic Definitions

Definition 2.1. A *topology* τ_X on a set X is a collection of subsets of X called *open sets*, which is closed under finite intersections and arbitrary unions. A *topological space* is a pair (X, τ_X) where τ_X is a topology on X. A *neighborhood* of $x \in X$ is a set S such that there is $U \in \tau_X$ with $x \in U \subseteq S$. $\lozenge$

Notation 2.2. We shall often refer to a topological space as just X, if it is unambiguous what the topology is. Extending this, any subset $S \subseteq X$ is by default equipped with the subspace topology

$$U \in \tau_S \overset{\Delta}{\iff} \exists U' \in \tau_X . U = U' \cap S$$

unless otherwise mentioned. We will refer to the closed sets of X by σ_X, the set of open neighborhoods at $x \in X$ by $\tau_X(x)$, and the set of neighborhoods by $N_X(x)$. $\lozenge$

Definition 2.3. Let S be a subset of a topological space X. The *interior* of S is the open set $\mathring{S} := \bigcup \{ U \in \tau_X \mid U \subseteq S \}$. The *closure* of S is the closed set $\overline{S} := \bigcap \{ C \in \sigma_X \mid S \subseteq C \}$. $\lozenge$

Definition 2.4. A space X satisfies the T_0 *separation condition* if for any two distinct points $x \neq y \in X$, there is an open set U such that either $x \in U \not\ni y$ or $x \notin U \ni y$. X is *Hausdorff* if for any two distinct points $x \neq y$, there are neighborhoods U_x and U_y of x and y respectively, such that U_x and U_y are disjoint. $\lozenge$

Definition 2.5. A *basis* for a space X is a family $\mathcal{B} \subseteq \tau_X$ of *basic open sets* such that every open is a union of basic opens. A *subbasis* for X is a family $\mathcal{B} \subseteq \tau_X$ of *subbasic open sets* such that every open is a union of finite intersection of subbasic opens. The (sub)basis $\mathcal{B}$ is *countable* if $\mathcal{B}$ is countable.

A *local basis* for a point $x \in X$ is a family $\mathcal{B}(x) \subseteq \tau_X(x)$ such that every neighborhood of x contains some $B \in \mathcal{B}(x)$. A local basis $\mathcal{B}$ for X assigns to each $x \in X$ a local basis at x. The local basis $\mathcal{B}$ is *countable* if $\mathcal{B}(x)$ is countable for each $x \in X$. $\lozenge$

Remark 2.6. A family $\mathcal{B} \subseteq \tau_X$ is a subbasis iff its saturation by finite intersections gives a basis. Any basis $\mathcal{B}$ gives a local basis by $\mathcal{B}(x) := \{ B \in \mathcal{B} \mid x \in B \}$. Moreover, all of these translations between the different notions preserve countability. $\circledcirc$

Limits & Sequentiality

In a space with a countable local basis, ω-sequences "suffice to describe the topology". This is captured by the following condition.

Definition 2.7. Let X be a topological space. A sequence $(x_i)_{i<\omega}$ of elements in X *converges* to x if for any $U \in \tau_X(x)$, U contains cofinitely many elements of $(x_i)_i$. We denote this by $(x_i)_i \to x$.

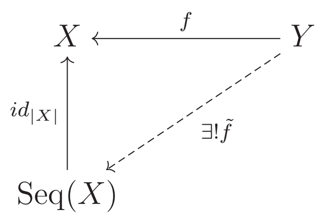
A set $U \subseteq X$ is *sequentially open* if for any converging ω-sequence $(x_i)_i \to x \in U$, the set U contains cofinitely many elements of $(x_i)_i$.

X is *sequential* if every sequentially open set is open. Let Seq denote the category of sequential spaces with continuous maps between them. $\lozenge$

Notation 2.8. Given a space X, the family of sequentially open sets in X is a topology $\mathrm{Seq}(\tau_X)$ which contains τ_X. We will denote the space $(X, \mathrm{Seq}(\tau_X))$ by $\mathrm{Seq}(X)$. $\lozenge$

Lemma 2.9 ([Sch02, Lemma 6])**.** *Let X and Y be topological spaces.*

1. *For a sequence $(x_i)_i$ and a point x in X, $(x_i)_i \to x$ in X iff $(x_i)_i \to x$ in $\mathrm{Seq}(X)$.*
2. *$\mathrm{Seq}(X)$ is sequential, i.e. $\mathrm{Seq}(\mathrm{Seq}(X)) = \mathrm{Seq}(X)$.*
3. *Every continuous map $f : X \to Y$ is also a continuous map $f : \mathrm{Seq}(X) \to \mathrm{Seq}(Y)$.*
4. *$\mathrm{Seq}(X)$ has the following universal property: for every continuous map $f : Y \to X$ from a sequential space Y, there is a unique continuous map $\tilde{f} : Y \to \mathrm{Seq}(X)$ making commute*



Corollary 2.10. Seq *is a coreflective subcategory of* Top*, i.e. sequentialization defines a faithful functor* $\mathrm{Seq} : \mathsf{Top} \to \mathsf{Seq}$ *which is right adjoint to the inclusion functor* $i : \mathsf{Seq} \to \mathsf{Top}$*.*

Lemma 2.11. *If a space X has a countable local basis, then X is sequential.*

Proof. Without loss of generality, we can assume the countable local base to be enumerated in descending order, i.e. for a point x, we can enumerate its basis $B_0, B_1, \ldots$ in such a way that $i \leq j$ implies $B_i \supseteq B_j$. This is because given any countable local basis $(C_i)_i$, we can take $B_0 = C_0$ and $C_{n+1} = B_{n+1} \cap C_n$.

So now consider a sequentially open set $S \subseteq X$. We need to show S is open, so it suffices to show for each $x \in S$, there is a local basic open B_i s.t. $x \in B_i \subseteq S$. Suppose for contradiction that this was not the case. Then for each local basic open $x \in B_i$, there is an element $x_i \in B_i - S$. This forms a sequence of elements $(x_i)_i$, and it converges to x since for any open set $x \in U$, we can find $x \in B_i \subseteq U$ for some B_i. Hence, $x_i \in U$ and since

$B_i \supseteq B_j$ for all $j \geq i$, we also have that $x_j \in U$. This shows that cofinitely many x_i are in U, i.e. that $(x_i)_i \to x$. Now, since S is sequentially open, we can find some $x_i \in S$, a contradiction. □

Lemma 2.12. *Let X be a sequential space and Y be a quotient of X. Then Y is sequential.*

Proof. Consider the quotient map $f : X \to Y$. Then the map $f : X = \mathrm{Seq}(X) \to \mathrm{Seq}(Y)$ is continuous. But that means

$$U \in \tau_{\mathrm{Seq}(Y)} \implies f^{-1}[U] \in \tau_X \implies U \in \tau_Y$$

so every sequentially open set in Y is open, i.e. $\mathrm{Seq}(Y) = Y$. □

Compactness

Definition 2.13. A space X is *compact* if for any family of open sets $\{ U_i \}_{i \in I}$ such that $X = \bigcup_{i \in I} U_i$, there is a finite subfamily $U_{i_1}, \ldots, U_{i_k}$ such that $X = U_{i_1} \cup \ldots \cup U_{i_k}$. A subset $S \subseteq X$ is *compact* if S is compact as a subspace. ◊

Definition 2.14. A space X is *locally compact* if every point $x \in X$ has a compact neighborhood K. ◊

Theorem 2.15 ([Eng89, Theorem 3.3.2])**.** *for every compact subset A of a locally compact Hausdorff space X and open set $V \supseteq A$, there is an open set U such that $\overline{U}$ is compact and $A \subseteq U \subseteq \overline{U} \subseteq V$.*

Function Spaces

The category of topological spaces is not cartesian closed, because there may be no good topology on the set of continuous functions $X \to Y$. However, if the domain space X is locally compact and Hausdorff, then we can use the following compact-open topology to obtain a function space with the appropriate universal property. This will suffice for our later algebraic topological concerns, since we are mostly concerned with function spaces involving locally compact and Hausdorff spaces.

Definition 2.16. Let X, Y be topological spaces. The function space $[X \to Y]$ is the set of continuous functions $X \to Y$ equipped with the compact-open topology generated by subbasic opens of the form

$\mathsf{M}(K, U) := \{ f : X \to Y \mid f[K] \subseteq U \}$ where $K \subseteq X$ is compact and $U \in \tau_Y$. ◊

Theorem 2.17. *If X is a Hausdorff, locally compact space and Y is a space, then $[X \to Y]$ is the exponential object in* Top.

Proposition 2.18. *If Y is T_0, then so is $[X \to Y]$.*

Proof. Suppose $f \neq g : X \to Y$, so there is $x \in X$ with $f(x) \neq g(x)$. Then since Y is T_0, we can find U containing $f(x)$ but not $g(x)$. The subbasic open $\mathsf{M}(\{x\}, U_x)$ then contains f but not g. □

Theorem 2.19 ([Eng89, Theorem 3.4.16])**.** *If X is a Hausdorff, locally compact space and Y is a space, with both having countable bases, then $[X \to Y]$ also has a countable base.*

2.2 Domain Theory

An important model of computation in this thesis is Scott's graph model of the lambda calculus, which is best understood in the context of *domain theory*. In this section, we introduce the graph model following a brief introduction to domain theory. We refer to [AJ95] for a much more comprehensive overview, and to [Vic96] for intuition on the topological perspective.

Directed-complete Posets (DCPOs)

Definition 2.20. Let $(X, \sqsubseteq)$ be a partially ordered set. A non-empty subset $S \subseteq X$ is *directed* if for every $x, y \in S$ there is $z \in S$ s.t. $x \sqsubseteq z$ and $y \sqsubseteq z$. ◊

Definition 2.21. A poset $(X, \sqsubseteq)$ is *directed-complete* if every directed subset $S \subseteq X$ has a least upper bound or *directed join* denoted $\bigsqcup S$, and X has a least element $\bot$. We will refer to a directed-complete poset by the acronym DCPO. ◊

Definition 2.22. Let X be a DCPO. An element $x \in X$ is *compact* if for any directed subset $S \subseteq X$, whenever $x \sqsubseteq \bigsqcup S$, there is some $y \in S$ s.t. $x \sqsubseteq y$. Let $\downarrow\!\!X$ denote the set of compact elements in X, and write $x \ll y$ to mean x is compact and $x \sqsubseteq y$. X is *algebraic* if for every element $x \in X$, the set

$$\downarrow\!\! x := \{ c \sqsubseteq x \mid c \text{ compact} \}$$

of compact elements below x is directed, and $x = \bigsqcup \Downarrow x$. ◊

Remark 2.23. The $\ll$ relation is usually known as the *way-below* relation. The usual definition of the way-below relation is more general, but coincides with the one we give above in the context of algebraic DCPOs. ⊚

Definition 2.24. A function $f : X \to Y$ between DCPOs is *monotone* if for all $x, x' \in X$, $x \sqsubseteq_X x'$ implies $f(x) \sqsubseteq_Y f(x)$. It is *Scott-continuous*, if it is monotone and additionally for any directed subset $S \subseteq X$, we have $f(\bigsqcup S) = \bigsqcup f[S]$. ◊

For algebraic DCPOs, since every element is a directed limit of compact elements, the behavior of a Scott-continuous map on compact elements determines the whole map.

Lemma 2.25. *Let X be an algebraic DCPO, and Y be a DCPO. Every monotone map $f : \Downarrow X \to Y$ has a unique Scott-continuous extension $\tilde{f} : X \to Y$.*

Scott Topology

Every DCPO comes equipped with a natural topology, the Scott topology, allowing us to view them as spaces.

Definition 2.26. Let $(X, \sqsubseteq)$ be a DCPO. Then a subset $U \subseteq X$ is *Scott-open* if it is upwards closed, and is furthermore inacessible by directed joins in the sense that for any directed set S, if $\bigsqcup S \in U$ then $S \cap U$ is non-empty. ◊

Remark 2.27. Thinking of directed sets as a generalization of increasing sequences, we can view a Scott-open set as some kind of sequentially open set, but only with respect to (generalized) increasing sequences. ⊚

Notation 2.28. The family of Scott-open sets in a DCPO form a topology called the *Scott topology*. When we refer to the topology τ_X of a DCPO X, we will always mean the Scott topology. ◊

The choice of this topology is justified in that it makes the Scott-continuous maps precisely the continuous maps.

Proposition 2.29. *Let X, Y be DCPOs. Then a function $f : X \to Y$ is Scott-continuous iff it is topologically continuous.*

For algebraic DCPOs, the Scott topology has an alternate, more intuitive definition, which says that a set U is Scott-open if it is "compactly observable".

Proposition 2.30. *Let X be an algebraic DCPO. Then $U \subseteq X$ is Scott-open iff it is upwards-closed and whenever $x \in U$, there is $x' \ll x$ such that $x' \in U$.*

Scott's Graph Model of λ-calculus

In [Sco76], Dana Scott introduced his Graph model of the λ-calculus. It is so called because it is a domain $\mathbb{P}$ such that the graph of any continous map $\mathbb{P} \to \mathbb{P}$ may be encoded as an element of $\mathbb{P}$. For this thesis, we are interested in realizability structures over $\mathbb{P}$. As we will see in the next chapter, the topology inherent in $\mathbb{P}$ imbue realizability structures over it with a topological flavour.

Definition 2.31. *Scott's graph model* $\mathbb{P} := (\mathcal{P}\,\mathbb{N}, \subseteq)$ is the powerset lattice over $\mathbb{N}$, considered as a DCPO. ◊

Remark 2.32. The compact elements of $\mathbb{P}$ are precisely the finite subsets of $\mathbb{N}$, which makes $\mathbb{P}$ an algebraic DCPO since every subset is a (directed) union of its finite subsets. The Scott topology $\tau_{\mathbb{P}}$ has a countable basis given by open sets $\uparrow x$ for any finite set $x \ll \mathbb{N}$.

Notice that by Lemma 2.25, the behavior of a continuous map $f : \mathbb{P} \to \mathbb{P}$ is determined by its behavior on the finite elements of $\mathbb{P}$, and therefore can be coded by an element of $\mathbb{P}$, as in the following. ⊚

For the remainder of this section let us fix bijective functions

$$\langle -, - \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad \text{and} \qquad \mathsf{fin} : \downarrow\!\!\!\!\cdot\, \mathbb{N} \to \mathbb{N}$$

coding pairs of natural numbers and coding finite subsets of natural numbers respectively. Using these, we now describe the encoding (and decoding) of continuous maps as elements of $\mathbb{P}$.

Definition 2.33.

1. Let $f : \mathbb{P} \to \mathbb{P}$ be a Scott-continuous map. We define the *graph* of F, denoted Γf, as an element of $\mathbb{P}$ given by

$$\Gamma f = \left\{ \langle n, m \rangle \in \mathbb{N} \,\middle|\, m, n \in \mathbb{N} \text{ and } m \in f(\mathsf{fin}^{-1}(n)) \right\}$$

2. Let $x \in \mathbb{P}$. We define the *encoded function* $\Lambda x : \mathbb{P} \to \mathbb{P}$ by

$$y \mapsto \left\{ m \in \mathbb{N} \,\middle|\, \exists n \in \mathbb{N}.\mathsf{fin}^{-1}(n) \subseteq y \text{ and } \langle n, m \rangle \in x \right\}$$

◊

Proposition 2.34. *For a Scott-continuous map $f : \mathbb{P} \to \mathbb{P}$, $\Lambda\Gamma f = f$. For an element $x \in \mathbb{P}$, Λx is continuous and $x \subseteq \Gamma\Lambda x$.*

This allows us to view an element $x \in \mathbb{P}$ as a continuous function $x : \mathbb{P} \to \mathbb{P}$ and vice versa, which is what allows $\mathbb{P}$ to model the untyped lambda calculus. In fact, it can model the untyped lambda calculus with a pairing operation, using the following encoding of pairs.

Proposition 2.35. *The function*

$$(x, y) \mapsto \{ 2n \mid n \in x \} \cup \{ 2n + 1 \mid n \in y \} : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$$

is a homeomorphism, which we will denote $\langle\langle -, - \rangle\rangle$.

Chapter 3

Equilogical and QCB Spaces

In this chapter, we introduce our main objects of study: the equilogical spaces and the QCB spaces. In the introduction, we motivated the study of equilogical spaces based on intuitions about what a computationally representable space is. In section 3.1, we validate these intuitions by introducing the equilogical spaces as a realizability structure over Scott's graph model, and in section 3.2 we study categorical constructions in the category of equilogical spaces.

As for QCB spaces, in section 3.3 we prove an equivalent definition in terms of more intrinsic properties of a QCB space. This intrinsic property allows us to reconstruct a representing equilogical space, which we prove in section 3.4. Finally, in section 3.5 we also study categorical constructions in the category of QCB spaces.

3.1 The Category Equ of Equilogical Spaces

Definition 3.1. Let ωT_0 denote the category of countably-based T_0 spaces. An *equilogical space* $(X, \sim_X)$ is an ωT_0 space X equipped with an equivalence relation $\sim_X$. ♢

Notation 3.2. We will abuse notation and mostly refer to an equilogical space by its underlying space X. If no specific notation for the equivalence relation is specified, we will refer to its equivalence relation by the subscripted relation symbol $\sim_X$. We will also write $|X|$ if we need to disambiguate between X and its underlying space. ♢

This definition originates from considerations in realizability, specifically as a simplification of the definition of a *modest set* over Scott's graph model. One can think of a modest set as a set A along with an assignment, to each element $a \in A$, of some computational representations/implementations in $\mathbb{P}$.

Definition 3.3. A *modest set* over $\mathbb{P}$ is a pair (A, r) of a set A and an assigment $r : a \in A \mapsto \emptyset \subsetneq r_a \subseteq \mathbb{P}$ containing the elements of $\mathbb{P}$ which *realize* a, such that $a \neq a'$ implies $r_a \cap r_{a'} = \emptyset$, i.e. every element of $\mathbb{P}$ realizes at most one element of A. ♢

Another way to view this definition is that the elements of A are equivalence classes over some subspace of $\mathbb{P}$. However, the following theorem shows that $\mathbb{P}$ is a "universal" ωT_0 space, in the sense that subspaces of $\mathbb{P}$ precisely correspond to the ωT_0 spaces.

Theorem 3.4 (Embedding Theorem [Bau00, Theorem 1.1.2])**.**

1. *Every subspace of* $\mathbb{P}$ *is* ωT_0.
2. *Every* ωT_0 *space embeds into* $\mathbb{P}$.
3. *In fact, for any* ωT_0 *space* X*, there is a bijection between the set of embeddings* $X \hookrightarrow \mathbb{P}$ *and the set of enumerations of countable subbases for* X.

Proof. The correspondence between embeddings and bijections is given as follows. Given a countable enumeration $B : \mathbb{N} \to \mathcal{B}$ of a subbasis for X, the corresponding embedding $e_B : X \hookrightarrow \mathbb{P}$ is $e_B(x) := \{ n \in \mathbb{N} \mid x \in B_n \}$. On the other hand, an embedding $e : X \hookrightarrow \mathbb{P}$ determines an enumeration $B^e_n := \{ x \in X \mid n \in e(x) \}$. □

In light of this theorem, we see that every modest set (A, r) can be seen as an equivalence relation on the space $\{ x \in \mathbb{P} \mid \exists a.x \in r_a \}$, while every equilogical space X (fixing a subbase enumeration B) corresponds to a modest set on $\mathbb{P}$ obtained by quotienting $e_B[X]$. Next, we define the morphisms of equilogical spaces, based on the morphisms of modest sets.

Definition 3.5. A morphism $f : A \to B$ of modest sets over $\mathbb{P}$ is a function $f : A \to B$ for which there exists an element $x \in \mathbb{P}$ which realizes this function. This means that $\Lambda x(r_a) = r_{f(a)}$ for each $a \in A$. ♢

Translating A and B to equilogical spaces X and Y, a realizer for the function f is a continuous map $F : \mathbb{P} \to \mathbb{P}$ such that restricted to X it induces a map $X \to Y$ which when quotiented induces f:

$$
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{F} & \mathbb{P} \\
\uparrow & & \uparrow \\
X & \xrightarrow{F|_X} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow[F|_X]{} & Y \\
\downarrow & & \downarrow \\
A \cong X/_{\sim_X} & \xrightarrow{f} & Y/_{\sim_Y} \cong B
\end{array}
$$

However, the following theorem shows that any continuous map between ωT_0 spaces actually extend into a map $\mathbb{P}$, so there is no need to exhibit the whole F, only a continuous map $X \to Y$.

Theorem 3.6 (Extension Theorem [Bau00, Theorem 1.1.3])**.** *Every continous map $f : X \to Y$ between subspaces X, Y of $\mathbb{P}$ has a continuous extension $F : \mathbb{P} \to \mathbb{P}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\mathbb{P} & \xrightarrow[F]{} & \mathbb{P}
\end{array}
$$

This means that a morphism between equilogical spaces is really just an equivalence-respecting continuous map, but identified only up to the function they represent under quotienting.

Definition 3.7. Let X and Y be equilogical spaces.

1. A continuous map $f : X \to Y$ is *equivariant* if

$$x \sim_X x' \implies f(x) \sim_Y f(x').$$

2. Two continuous equivariant maps $f, g : X \to Y$ *represent the same map*, denoted $f \sim_{X \to Y} g$, if for all $x \in X$,

$$f(x) \sim_Y g(x).$$

3. A *morphism of equilogical spaces* is a $(\sim_{X \to Y})$-equivalence class of maps.

4. Let Equ be the category of equilogical spaces with morphisms as defined above.

◊

3.2 Categorical Constructions in Equ

The limits in Equ are computed as one would expect for topological spaces, but with identification up to equality replaced with identification up to equivalence. As for colimits, for topological spaces these are generally computed by quotienting/gluing. For equilogical spaces however, we perform the gluing "formally", i.e. by adding additional equivalences and leaving the underlying space untouched.

Theorem 3.8. Equ *has countable limits & colimits.*

Proof.

1. (products) The countable product $\Pi_{i\in I}(X_i, \sim_i)$ is the product[1] of underlying spaces equipped with the product of equivalence relations $(\Pi_i X_i, \Pi_i \sim_i)$.

2. (equalizer) Consider $[f], [g] : X \rightrightarrows Y$. Then we can take the equalizer to be the subset $E = \{\, x \in X \mid f(x) \sim_Y g(x) \,\}$, inheriting the subspace topology and the sub-equivalence relation.

3. (coproduct) The countable coproduct can be obtained by taking coproducts of the underlying spaces and coproducts of the equivalence relations.

4. (coequalizer) Consider $[f], [g] : X \rightrightarrows Y$. The coequalizer takes the underlying space $Q = Y$, but equipped with the least equivalence relation $\sim_Q$ containing both $\sim_Y$ and

$$y_1 \sim y_2 \iff \exists x \in X. y_1 = f(x) \text{ and } y_2 = g(x).$$

□

[1] Note that the countable product still has a countable basis since a basic open of $\Pi_{i\in I} X_i$ is a product of opens in each X_i of which *only finitely* many are allowed to be non-trivial.

The category Equ comes with one surprising feature in that it is cartesian closed, which is somewhat unexpected since we added equivalence relations in order to fix closure under quotients, not exponentials. On the other hand, it is less surprising from the modest set perspective, but directly translating the construction of exponentials for modest sets to Equ leads to a somewhat unintuitive construction. We refer to [Bau00, Theorem 4.1.5] for more details. However, in case the underlying spaces already have an exponential in ωT_0, the computation does become quite intuitive, as we now demonstrate.

Proposition 3.9 ([Bau00, Proposition 4.1.7])**.** *If X and Y are equilogical spaces and $W \in \omega T_0$ is an exponential*[2] *of the underlying spaces X and Y, then the exponential equilogical space Y^X is the equilogical space $(V, \equiv_W)$, where $\equiv_W$ is the partial equivalence relation on W defined by*

$$f \equiv_W g \iff \forall x, x'. x \sim_X x' \implies \mathrm{ev}(f, x) \sim_Y \mathrm{ev}(g, x')$$

and $V = \mathrm{dom}(\equiv_W)$.

In order to use this special case, it is natural to ask: when do weak exponentials exist in ωT_0? For our later homotopy-theoretic endeavours, we do not need a full characterization, just a sufficient condition that encompasses our use cases. Luckily, the function space construction in Top for locally compact Hausdorff exponents preserves ωT_0-ness.

Lemma 3.10. *If X is a locally compact Hausdorff ωT_0 space, and Y is an ωT_0 space, then the function space $[X \to Y]$ with the compact-open topology (Definition 2.16) is an ωT_0 space.*

Proof. By Theorem 2.17, Proposition 2.18 and Theorem 2.19. □

To summarize, we obtain the following construction of function spaces in Equ.

Corollary 3.11. *If X and Y are equilogical spaces with X having an underlying locally compact Hausdorff space, then the equilogical space with underlying space being the subspace $[X \to_{\mathsf{Equ}} Y] \subseteq [X \to Y]$ of equivariant maps and with the equivalence relation $\sim_{X \to Y}$ is the exponential Y^X in* Equ*.*

[2] in fact, it suffices for W to be a *weak* exponential.

3.3 Quotients of Countably-based (QCB) Spaces

In the previous section, we obtained a basic understanding of an equilogical space as a set A with an implementation or realization in $\mathbb{P}$. Since this implementation is a surjective map from a subspace of $\mathbb{P}$, we can induce the quotient topology on the set, and thus view it as a topological space. In this section, we investigate the category QCB of topological spaces that arise in this way by giving it a more intrinsic characterization.

The work in this section follow the line of work on admissible representations by Matthias Schröder, Ingo Battenfeld & Alex Simpson [BSS07; Sch02; Sch16; Sch21]. However, while their work realizes spaces using a model of computation known as Kleene's second model $\mathbb{N}^\omega$, we are interested in the connection with $\mathbb{P}$ instead. Therefore, we replicate some of their arguments, replacing $\mathbb{N}^\omega$ with $\mathbb{P}$. It is not so surprising that we can do this since $\mathbb{N}^\omega$ is itself an ωT_0 space and so must embed into $\mathbb{P}$. In chapter 7, we explain the tradeoffs following from this switching of computational models.

Definition 3.12.

1. The quotienting functor $L : \mathsf{Equ} \to \mathsf{Top}$ sends an equilogical space X to the T_0 quotient $(X/_{\sim_X})/_0$, and morphisms $[f] : X \to Y$ to the map induced by the universal property of quotients. This definition is coherent since two maps f, g represent the same morphism precisely when they induce the same map on quotients $X/_{\sim_X} \to Y/_{\sim_Y}$, which means they must also induce the same map on the T_0 quotient.

2. A topological space X is QCB if it is in the (essential) image of L. Let QCB denote the full subcategory of Top containing the QCB spaces.

◊

Remark 3.13. The quotienting functor restricts to a functor $L : \mathsf{Equ} \to \mathsf{QCB}$, but it is not full since not every map of quotient spaces can be induced by a map on the original spaces. ⊚

To begin with, we can simplify this definition of QCB to something more workable.

Lemma 3.14. *Let X be a T_0 topological space. Then it is a quotient of a countably-based space iff it is a quotient of an ωT_0 space.*

Proof. Clearly, the right-to-left-direction holds. For the other direction, suppose we have a T_0 quotient space $Y/_\sim$ of a countably-based space Y. Let $q : Y \to Y/_\sim$ denote the quotient map. Unwrapping the T_0 property on $Y/_\sim$, we have that for any $y_1, y_2 \in Y$,

$$\Big(\forall U \in \tau_{Y/\sim}.y_1 \in q^{-1}[U] \iff y_2 \in q^{-1}[U]\Big) \implies y_1 \sim y_2$$

Let $y_1 \sim_{q,0} y_2$ denote the relation in the antecedent of the above statement.

We will establish a homeomorphism $(Y/_\sim) \cong (Y/_0)/_{\sim'}$ where

$$q_0(y_1) \sim' q_0(y_2) \overset{\Delta}{\iff} y_1 \sim y_2.$$

Note that this definition is coherent due to the T_0 condition on $Y/_\sim$. That is, if we have $q_0(y_1) = q_0(y_1')$ and $q_0(y_2) = q_0(y_2')$, then $y_1 \sim y_2$ iff $y_1' \sim y_2'$. We will denote the quotient map for $\sim'$ by $q' : Y/_0 \to (Y/_0)/_{\sim'}$.

The homeomorphism is the map $f : (Y/_\sim) \to (Y/_0)/_{\sim'}$ induced by the universal property of the quotient $Y/_\sim$ on the composite $q'q_0$. Explicitly, the map sends $q(y)$ to $q'q_0(y)$. It is clear that f is surjective and continuous, so it remains to show that f is injective and an open map.

(f is injective) Suppose $f(q(y_1)) = f(q(y_2))$. Then $q'q_0(y_1) = q'q_0(y_2)$. By definition of $\sim'$, this means $y_1 \sim y_2$, so $q(y_1) = q(y_2)$.

(f is open) Suppose $U \subseteq Y/\sim$ is open, AKA $q^{-1}[U] \subseteq Y$ is open. We need to show $f[U]$ is open, AKA $q_0^{-1}[q'^{-1}[f[U]]] \subseteq Y$ is open. It suffices to show $q_0^{-1}[q'^{-1}[f[U]]] = q^{-1}[U]$, as follows.

$$\begin{aligned}
&y \in q_0^{-1}[q'^{-1}[f[U]]] \\
\iff &q'(q_0(y)) \in f[U] \\
\iff &f^{-1}(q'(q_0(y))) \in U \text{ by injectivity of } f \\
\iff &q(y) \in U \\
\iff &y \in q^{-1}[U]
\end{aligned}$$

□

Proposition 3.15. *A space X is QCB iff it is T_0 and a quotient of some countably-based space.*

Proof. If X is QCB, then it is by construction T_0 and a quotient of a countably-based space. On the other hand, if X is T_0 and a quotient of some countably-based space, then by the previous lemma we can consider it as a quotient $(X'/_\sim)$ of an ωT_0 space X'. Then since $X \cong X'/_\sim$ is already T_0, $L(X', \sim) = (X'/_\sim)/_0 \cong X'/_\sim \cong X$. □

We now seek the intrinsic characterization of QCB spaces that does not rely on quotienting other spaces. Countable bases are not preserved under quotienting, but the weaker condition of *sequentiality* (Definition 2.7) is preserved under quotienting (Lemma 2.12). Therefore, any QCB space will be sequential. However, to see exactly what aspects of a basis is preserved under quotienting, we have to more closely inspect the notion of basis in terms of sequences.

Let X be a sequential space with a basis. Then a set U is open iff U cofinitely covers every sequence converging into U iff every sequence converging into U is cofinitely covered by some basic subset $B \subseteq U$. This works because B is open and so cofinitely covers *every* sequence. However, in order to "perform its job" of generating U, it only needs to cofinitely cover a given sequence converging in U. The notion of basis is in this sense overqualified for its job, and so we consider the weaker notion of *pseudobasis*, which only exactly does its job. This notion is rather more robust and in particular does survive under quotienting[3]. Moreover, the countability of the pseudobase is preserved under quotienting. As we will soon see, the sequentiality and countable pseudobase suffice as a characterization of the properties of ωT_0 spaces that survive under quotienting.

Definition 3.16. Let X be a topological space. A *pseudobase* $\mathcal{B}$ for X is a set of subsets of X such that for every x in some open set U, and a sequence $(x_i)_i \to x$, there is a pseudobase $B \in \mathcal{B}$ such that $x \in B \subseteq U$ and B contains cofinitely many elements of the sequence $(x_i)_i$. ♢

Example 3.17. Any basis for a space X is also a pseudobase. ♢

Pseudobases are a weaker notion, but they are more robust than bases in that they are preserved under more constructions, in addition to the usual preservation properties of bases.

Proposition 3.18. *Let X be a space with a countable pseudobase.*

1. *Any subspace of X also has a countable pseudobase.*

[3] This narrative is nice but not exactly true, for the construction of the pseudobase in the quotient space is not just the quotient-image of the original pseudobase, and is a fair bit more complicated. We refer to the proof of [Sch16, Lemma 3.1.12] for more details.

2. *A countable topological product of spaces with countable pseudobase also has a countable pseudobase.*

3. *If X is sequential and Y is a quotient of X, then Y is also sequential (Lemma 2.12) with a countable pseudobase [Sch16, Lemma 3.1.12].*

4. $\mathrm{Seq}(X)$ *also has a countable pseudobase [Sch16, Lemma 3.1.11].*

5. *If X is sequential and Y is another sequential space with a countable pseudobase, then* $\mathrm{Seq}([X \to Y])$ *has a countable pseudobase [Sch16, Proposition 4.2.7 (3)].*

Corollary 3.19. *Let X be a QCB space. Then it is sequential and has a countable pseudobase.*

We next show that the properties of T_0, sequentiality and having a countable pseudobase is enough to characterize the QCB spaces. This is done by constructing, from the data of the pseudobase, an equilogical space $R(X)$ which when quotiented gives X. Essentially, an enumeration of a pseudobase for a space X allows us build a $\mathbb{P}$-modest set by realizing each $x \in X$ by (numerical encodings of) its families of pseudobasic neighborhoods. Sequentiality ensures the data of the topology of X is indeed fully captured by the pseudobase, and so the quotient space constructed from our $\mathbb{P}$-modest set is the same as the topology on X. The construction is adapted from [Sch02, Theorem 12].

Definition 3.20. Let X be a T_0 sequential space, with an enumeration of a countable pseudobase $B : \mathbb{N} \to \mathcal{B}$ for X. Its associated $\mathbb{P}$-realizability relation is defined as

$$\begin{aligned} p \vdash_{X,B} x \overset{\Delta}{\iff} & \forall n \in p. x \in B_n \\ & \text{and } \forall U \in \tau_X.(x \in U \implies \exists n \in p. x \in B_n \subseteq U). \end{aligned}$$

◊

Remark 3.21. Note that this is a realizability relation because every element $x \in X$ is realized by $p_x := \{ n \in \mathbb{N} \mid x \in B_n \}$. ⊚

Theorem 3.22. *Let X be a T_0 sequential space, with an enumeration of a countable pseudobase $B : \mathbb{N} \to \mathcal{B}$ for X. Then $\vdash_{X,B}$ is a modest relation, thereby inducing an equilogical space $R_B(X)$. Moreover, the canonical map $L(R_B(X)) \to X$ is a homeomorphism.*

Proof. For this proof, we will omit the subscript and refer to $\vdash_{X,B}$ as $\vdash$.

($\vdash$ is modest) Suppose $p \vdash x$ and $p \vdash y$, but $x \neq y$. Since X is T_0, WLOG we have an open set U s.t. $x \in U$ and $y \notin U$. Then by definition of $p \vdash x$, there is $n \in p$ such that $x \in B_n \subseteq U$. However, $p \vdash y$ means that $y \in B_n \subseteq U$, a contradiction. Therefore, $x = y$.

Since we have a modest realizability relation, this induces an equilogical space $RX = (P, \sim_P)$ where

$$P = \{\, p \in \mathbb{P} \mid \exists x \in X.p \vdash x \,\}$$

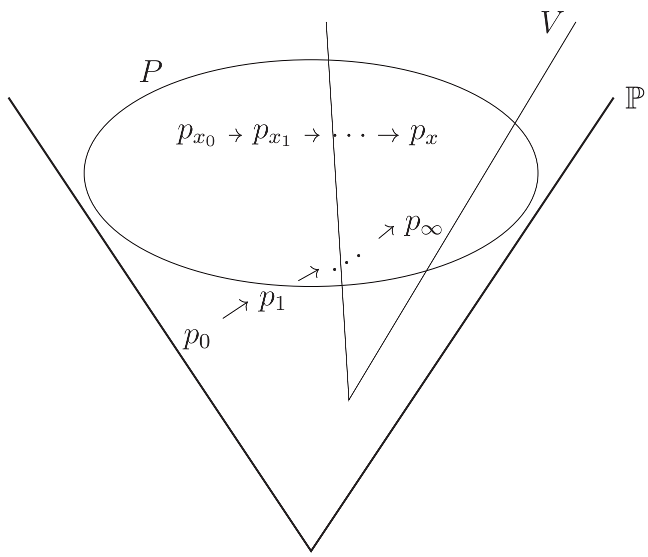
$$p \sim_P p' \iff \exists x \in X.p \vdash x \text{ and } p' \vdash x$$

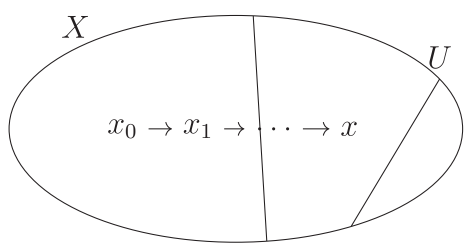
The realizability relation is then a surjective function $q : P \to X$ mapping $p \in P$ to the unique $x \in X$ it realizes. We will now show this map is a quotient map, which makes the induced map $P/_{\sim_P} \to X$ a homeomorphism.

(U open $\implies q^{-1}[U]$ open) It suffices to show for any $p \in q^{-1}[U]$, that there exists an open V s.t. $p \in V \subseteq q^{-1}[U]$. By definition of q, we have $p \vdash q(p) \in U$, so there is some $n \in p$ such that $q(p) \in B_n \subseteq U$. Then taking the desired open set to be $V := \uparrow\{\, n \,\} \cap P$, we have that $p \in V$. We also have $V \subseteq q^{-1}[U]$ because for any $p' \in V$,

$$n \in p' \implies q(p') \in B_n \subseteq U \implies p' \in q^{-1}[U].$$

($q^{-1}[U]$ open $\implies U$ open) Suppose $q^{-1}[U]$ is open, so $q^{-1}[U] = P \cap V$ for some Scott-open set V in $\mathbb{P}$. We now try to show U is sequentially open, so let (x_i) be a sequence that converges to $x \in U$. The idea is to use the sequence $(p_{x_i})_i \to p_x$ in P that maps down to $(x_i)_i \to x$, and then use the Scott-open property of V to show it has to contain cofinitely many p_{x_i}, and then conclude that U contains cofinitely many x_i. The problem of course is that this sequence $(p_{x_i})_i$ is not an increasing sequence, which we need to invoke the Scott-open property. To resolve this, we instead construct an increasing sequence in $\mathbb{P}$ that approximates $(p_{x_i})_i$.





We define

$$p_i := \{\, n \in \mathbb{N} \mid \exists U \in \tau_X.\{x_i, x_{i+1}, \ldots, x\} \subseteq B_n \subseteq U \,\},$$

from which we obtain the set

$$p_\infty := \bigcup_{i \in \mathbb{N}} p_i = \{\, n \in \mathbb{N} \mid \exists U \in \tau_X.\exists i \in \mathbb{N}.\{x_i, x_{i+1}, \ldots, x\} \subseteq B_n \subseteq U \,\}$$

We make some observations:

- The sequence is increasing: $p_i \subseteq p_{i+1}$.
- The sequence approximates $(p_{x_i})_i$, i.e. $p_i \subseteq p_{x_i}$.

- $p_\infty \in P$ with $q(p_\infty) = x$.

Hence, $p_\infty \in q^{-1}[U] = P \cap V$, and since V is inacessible by directed joins, there is some $i \in \mathbb{N}$ such that $\{p_i, p_{i+1} \ldots\} \subseteq V$. But since V is upwards closed, $\{p_{x_i}, p_{x_{i+1}}, \ldots\} \subseteq P \cap V = q^{-1}[U]$. This shows $\{x_i, x_{i+1}, \ldots\} \subseteq U$, i.e. cofinitely many x_i are in U. Hence, U is sequentially open and therefore open. $\square$

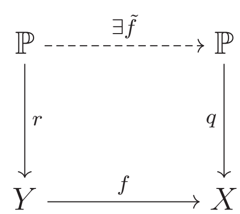
Corollary 3.23. *A space X is QCB iff it is a T_0 sequential space with a countable pseudobase.*

3.4 QCB as a reflective subcategory of Equ

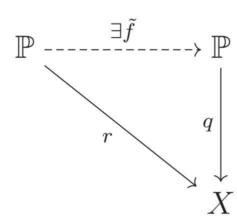
In the preceding section, we associated each QCB space X to a particular equilogical space $R_B(X)$. In this section, we will show that $R_B(X)$ is particularly well-behaved, in the sense that we have an adjunction

$$\frac{LY \to X}{\overline{Y \to R_B(X)}}.$$

Proposition 3.24 (Universality)**.** *Let X be QCB, and Y a space. Then the partial quotient map $q : \mathbb{P} \rightharpoonup X$ associated with $R_B(X)$ in Theorem 3.22 has the property that for any surjective partial continuous map $r : \mathbb{P} \rightharpoonup Y$, and every continuous map $f : Y \to X$, there is a continuous lift $\tilde{f} : \mathbb{P} \to \mathbb{P}$ making the square commute [BSS07, Proposition 3.12]:*



Equivalently, for every partial continuous function $r : \mathbb{P} \rightharpoonup X$, There is a continuous map $\tilde{f} : \mathbb{P} \to \mathbb{P}$ making the triangle commute [Sch21, Definition 9.3.4]:



Proof. First, we establish the equivalence between the two lifting properties given above. Assuming the square property, we can prove the triangle property by taking $f = \mathrm{im}(r) \hookrightarrow X$. Assuming the triangle property, we can prove the square property by taking the r in the triangle diagram to be the fr from the square diagram.

Let us now prove that q has the triangle property, adapting the proof from [Sch02, Theorem 12]. Let us begin by observing what properties such a lift $\tilde{f}$ should have. The commutativity requirement means that we have to show, for all $p \in \mathrm{dom}(r)$,

$$\tilde{f}(p) \vdash_{X,B} r(p).$$

Recall that this means we have to show

$$\forall n \in \tilde{f}(p).r(p) \in B_n \tag{3.1}$$

and for all $U \in \tau_X$,

$$r(p) \in U \implies \exists n \in \tilde{f}(p).r(p) \in B_n \subseteq U. \tag{3.2}$$

If $\tilde{f}$ is to be continuous, then defining $\tilde{f}$ on finite elements of $\mathbb{P}$ forces a definition for the infinite elements by $\tilde{f}(p) := \bigcup_{p' \ll p} \tilde{f}(p')$. We define $\tilde{f}$ by

$$\tilde{f}(p' \text{ finite}) := \left\{ \phi(k) \in \mathbb{N} \;\middle|\; B_{\phi(k)} \supseteq r[\uparrow p'] \right\}$$

where

$$\phi : \mathbb{N} \to \mathbb{N} : k \mapsto k - \left\lfloor \sqrt{k} \right\rfloor$$

i.e. it is the function which performs increasingly bigger loops:

k	0	1	2	3	4	5	6	7	8	9	...
$\phi(k)$	0	1	0	1	2	0	1	2	3	0	...

The reasoning behind this definition will become clear when we prove condition 3.2. First, let us note that condition 3.1 holds: if $n \in \tilde{f}(p) = \bigcup_{p' \ll p} \tilde{f}(p')$ then $n \in \tilde{f}(p')$ for some finite $p' \subseteq p$. By definition of $\tilde{f}(p')$, we have that $r(p) \in r[\uparrow p'] \subseteq B_n$.

Now, we verify condition 3.2. Suppose that $r(p) \in U$. Further unfolding the condition, we have to prove that there is $p' \ll p$ and $m \in \mathbb{N}$ such that $r[\uparrow p'] \subseteq B_{\phi(m)} \subseteq U$. Let us prove this by contradiction, so suppose the condition does not hold, i.e. for all $p' \ll p$ and $m \in \mathbb{N}$, if $B_{\phi(m)} \subseteq U$, there is some $p_{p',m} \in \uparrow p'$ such that $r(p_{p',m})$ is defined and $r(p_{p',m}) \notin B_{\phi(m)}$. If $B_{\phi(m)} \nsubseteq U$, let us anyway denote $p_{p',m} := p$. This way, $p_{p',m}$ is always defined under r and approximated by p'.

We will construct a sequence converging to p using this. First, since p is countable we can fix some enumeration of its contents:

$$p = \{ n_0, n_1, n_2 \dots \}$$

For $k \in \mathbb{N}$, let $[n_k] = \{ n_0, \dots n_k \} \ll p$. Then we can define a sequence

$$p_k := p_{[n_k],\phi(k)}$$

We make the following observations.

1. This sequence $(p_k)_{k \leq \omega}$ converges to p because it is approximated by the increasing sequence $[n_k]$ for which $\bigcup_{k \in \mathbb{N}} [n_k] = p$. Then by (sequential) continuity of r, the sequence $(r(p_k))_{k \leq \omega}$ converges to $r(p) \in U$.

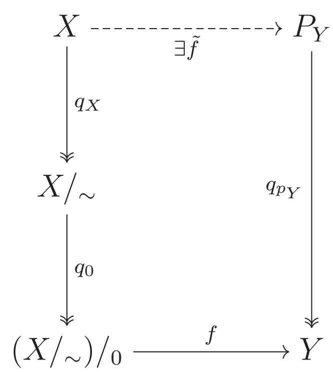
2. By the pseudobase property of X, there is some $m \in \mathbb{N}$ such that $r(p) \in B_m \subseteq U$ and B_m contains cofinitely many elements of the sequence $(r(p_k))_{k \leq \omega}$. Let $m_0 \in \mathbb{N}$ be such that $\{ r(p_{m_0}), r(p_{m_0+1}), r(p_{m_0} + 2), \dots r(p) \} \subseteq B_m$.

3. By our loopy definition of ϕ, $m = \phi(k)$ for some $k \geq m_0$. Hence, $r(p_k) = r(p_{[n_k],\phi(k)}) \notin B_{\phi(k)} = B_m$. But this contradicts the previous point, which claims $r(p_k) \in B_m$.

□

Corollary 3.25. *Let $X \in \mathsf{Equ}$, and $Y \in \mathsf{QCB}$ with a pseudobase enumeration B. For any map $f : LX \to Y$, there is a unique lift $[\tilde{f}] : X \to R_B(Y)$ such that $L[\tilde{f}] = f$.*

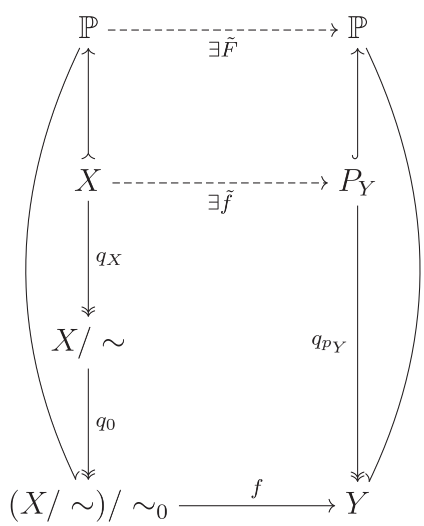
Proof. Unwrapping what this means, we have to show that for any map $f : ((X/_\sim)/_0) \to Y$, there is an equivariant map $\tilde{f}$, unique up to its induced quotient map, making the following diagram commute:



The uniqueness easily follows, since for any two such $\tilde{f}_1$ and $\tilde{f}_2$,

$$\begin{aligned} \tilde{f}_1 \sim \tilde{f}_2 &\iff \forall x. \tilde{f}_1(x) \sim_{P_Y} \tilde{f}_2(x) \\ &\iff \forall x. q_{P_Y} \tilde{f}_1(x) = q_{P_Y} \tilde{f}_2(x) \iff \forall x. f q_0 q_X(x) = f q_0 q_X(x) \end{aligned}$$

Hence, it remains to show existence. For this, note that X is an ωT_0 space, so by the embedding theorem it must embed into $\mathbb{P}$ once we pick some enumeration of a subbase for X. Then the quotient map $q_0 q_X$ is isomorphic to a partial quotient map on $\mathbb{P}$, and similarly for P_Y, leading to the scenario in the following diagram.



By universality, we can induce a filler map $\tilde{F}$, which restricts to the desired $\tilde{f}$. □

As a formal consequence of this lifting, and the fact that we have an isomorphism $LR_B(X) \cong X$, we can turn $R_B(X)$ into a fully faithful right adjoint of L, i.e. we can view QCB as a reflective full subcategory of the equilogical spaces.

Definition 3.26. Fix a pseudobase enumeration B for each QCB space X. The functor $R : \mathsf{QCB} \to \mathsf{Equ}$ maps each QCB space X to the equilogical space $R_B(X)$, and $f : X \to Y$ to the unique lift of the composite

$$LRX \overset{\varepsilon}{\cong} X \xrightarrow{f} Y \overset{\varepsilon}{\cong} LRY$$

$\Diamond$

Corollary 3.27. *R is a fully faithful right adjoint of L.*

3.5 Categorical Constructions in QCB

The characterization of QCB spaces as T_0 sequential spaces with a countable pseudobase also allows us to get a better grasp on the categorical constructions in QCB. The idea is that computing limits of QCB spaces as one would in Top generally breaks sequentiality, but preserves T_0-ness and the countable pseudobase. On the other hand, computing colimits as one would in Top breaks T_0-ness. Both of these problems are relatively easy to fix, by taking the sequentialization and doing T_0 quotienting. Of course, we can only do countable versions of these categorical constructions as otherwise we would break the countable pseudobase property.

Proposition 3.28. *Let $J : \mathcal{D} \to \mathsf{QCB}$ be a diagram with countably many objects. Let the composite be denoted $J' : \mathcal{D} \to \mathsf{QCB} \hookrightarrow \mathsf{Top}$. Then*

1. $\lim J \cong \mathrm{Seq}(\lim J')$
2. $\operatorname{colim} J \cong (\operatorname{colim} J')/_0$

Proof. 1. $\lim J'$ is (isomorphic to) a subspace of the cartesian product of spaces (with Tychonoff topology) in the diagram J. Hence, by Proposition 3.18, $\lim J'$ is T_0 and has a countable pseudobase. Then $\mathrm{Seq}(\lim J')$ is still T_0 with a countable pseudobase, but is additionally sequential and therefore QCB. Since $\mathrm{Seq}(-)$ is a right adjoint, $\mathrm{Seq}(\lim J')$ is limiting with respect to any sequential cone over J', and in particular over any QCB cone.

2. The proof is dual to the one for limits. $\operatorname{colim} J'$ is (isomorphic to) a quotient of the disjoint sum of spaces in the diagram J. Hence, by Proposition 3.18, $\operatorname{colim} J'$ is sequential and has a countable pseudobase. Then $(\operatorname{colim} J')/_0$ is a further quotient, so is T_0, sequential and has a countable pseudobase, i.e. a QCB space. Since the T_0-quotienting is a left adjoint, $(\operatorname{colim} J')/0$ is colimiting with respect to any T_0 cocone over J', and in particular over any QCB cocone.

□

For finite products, the sequentialization is actually unnecessary if one of the components of the product is locally compact. This will be useful in the next chapter, where we need to work with the cylinder space $X \times I$.

Theorem 3.29 ([ABM85, Theorem 4])**.** *Let X and Y be sequential spaces, and Y be locally compact. Then $X \times Y$ is sequential.*

Finally, exponentials of QCB spaces are given the sequential-open topology, but this still falls short of being sequential so we have to sequentialize it.

Definition 3.30. Let X and Y be topological spaces. The sequential-open topology $\tau_{\rightarrow}$ on Y^X is generated by subbasic opens of the form

$$C((x_i)_{i<\omega} \to x_\infty, V) := \{\, f : X \to Y \mid f\{\, x_0, x_1, x_2 \dots x_\infty \,\} \subseteq V \,\}$$

Where $(x_i)_{i<\omega}$ converges to x_∞ in X and $V \in \tau_Y$. ◊

Proposition 3.31 ([BSS07, Proposition 4.9])**.** *Let X and Y be QCB spaces. Then $\operatorname{Seq}(Y^X, \tau_{\rightarrow})$ is the exponential object in QCB.*

Observe that the subbasic opens of the form above are also subbasic opens in the compact-open topology. In [BSS07], it is remarked that the sequentialization of $\tau_{\rightarrow}$ is also the sequentialization of the compact-open topology, and so the topology of Y^X in QCB is finer than the compact-open topology.

Chapter 4

Structures for Homotopy Theory

In this chapter, we introduce various structures on a category for modelling homotopy theory, as well as the classical homotopy theory for topological spaces. Our aim in the next two chapters is to construct these structures on QCB and Equ. In the following, we work on an arbitrary category $\mathcal{C}$. When we define notions depending on the existence of certain categorical constructions in $\mathcal{C}$, we implicitly assume a category in which these constructions exist.

4.1 Interval Objects

Classically, homotopy theory is the study of when two maps or two spaces are equivalent up to *homotopy*.

Definition 4.1. Let X, Y be topological spaces. A *homotopy* $H : f \simeq g$ between maps $f, g : X \to Y$ is a map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. A map $f : X \to Y$ is a *homotopy equivalence* if there is a *homotopy inverse* map $g : Y \to X$ such that $fg \simeq id_Y$ and $gf \simeq id_X$. ♢

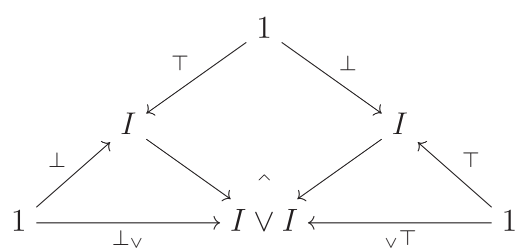
Homotopy formalizes the notion of continuous deformation of maps and spaces, without cutting. An important aspect of this definition is the presence of the unit interval $\mathbf{I} := [0,1]$, which really shapes the flow of deformation. That is, we formalize deformation as an $\mathbf{I}$-shaped movement of X-shapes inside Y.

I is special because it can be stretched and squished and twisted as desired. More specifically, there is an endpoint-preserving homeomorphism from **I** to two copies of **I** glued at the middle[1]: $\mathbf{I} \vee \mathbf{I} \cong [0, 2]$.

Another way to think of this is as an operation which cuts **I** into two **I**s. An immediate consequence of this operation is that homotopy becomes a transitive relation. This is rather intuitive: if we can deform f to g and g to h, then we can deform f to h. There is additionally an endpoint-permuting "flip" homeomorphism from **I** to itself, which makes homotopy into a symmetric relation.

Based on this, one way of abstracting homotopy theory to an arbitrary category is to simply designate an object as the interval. Of course, to ensure the symmetry and transitivity of the resulting homotopy, one must ensure this object has enough structure.

Definition 4.2. An *interval object* $(I, \bot, \top)$ in $\mathcal{C}$ is an object I equipped with two global elements $\bot, \top : 1 \to I$ called the *source* and *target* respectively. A morphism of interval objects $f : (I, \bot_I, \top_I) \to (J, \bot_J, \top_J)$ is a map $f : I \to J$ s.t. $f\bot_I = \bot_J$ and $f\top_I = \top_J$. A *flip map* or *twist map* for an interval object $(I, \bot, \top)$ is a map $\tau : (I, \bot, \top) \to (I, \top, \bot)$ such that $\tau\tau = id_I$. A *cut map* is a map $\kappa : (I, \bot, \top) \to (I \vee I, \bot_\vee, {}_\vee\top)$, where $I \vee I$ is the pushout:



◊

Definition 4.3. Let $\mathcal{C}$ be a category equipped with an interval object $(I, \bot, \top)$. An *I-homotopy* between maps $f, g : X \to Y$ is a map $H : X \times I \to Y$ such that $H(X \times s) = f$ and $H(X \times t) = g$, denoted $f \simeq_I g$ or just $f \simeq g$ if the I is obvious. A map $f : X \to Y$ is an I-homotopy equivalence if there is a map $g : Y \to X$ such that $gf \simeq id_X$ and $fg \simeq id_Y$. ◊

[1] In fact, **I** is even more special than this in that it is the *universal* such space X with two distinct points and an endpoint-preserving map into $X \vee X$ [Lei11]. This gives a coalgebraic perspective on **I** as the initial coalgebra of the "stretch" operation $X \mapsto X \vee X$.

Of course, equivalently a homotopy may be seen as a map $H : X \to Y^I$, if the exponential exists. While these two notions are formally the same when constructed using an exponentiable interval object, in the next section we explain that they arise from different conceptions of deformation. Before that, we prove that indeed an interval object with enough structure generates a congruent homotopy relation.

Proposition 4.4. *For an interval object* $(I, \bot, \top, \tau, \kappa)$*, I-homotopy is a congruence relation on maps in* $\mathcal{C}$*.*

Proof. Reflexivity, symmetry and transitivity follow from maps

$$r : X \cong X^1 \to X^I \qquad \tau : X^I \to X^I \qquad \mu : X^{I \vee I} \to X^I$$

induced by the exponential functor on $! : I \to 1$, τ and κ. For congruence with pre- and post-composition, if $H : f \simeq g$ then $h^I H : hf \simeq hg$ and $Hh : fh \simeq gh$ for appropriately composable h. □

In order to study morphisms in a category $\mathcal{C}$ up to a given homotopy congruence relation, we study its corresponding homotopy category.

Definition 4.5. Let $\simeq$ be a congruence relation on maps in $\mathcal{C}$. Then we define the *homotopy category* $\mathrm{Ho}_{\simeq}(\mathcal{C})$ induced by $\simeq$ as having the same objects as $\mathcal{C}$, but morphisms are equivalence classes of maps in $\mathcal{C}$. If $\simeq$ is induced by an interval I, then we will write $\mathrm{Ho}_I(\mathcal{C})$ instead. ◊

4.2 Cylinder & Cocylinder Objects

Instead of viewing deformations as an **I**-shaped movement of X-shapes in Y, another way to view deformations is as a continuous X-indexed family of **I**-shapes AKA paths in Y. With this view, one might rather regard a homotopy as a map $X \to Y^{\mathbf{I}}$. The space $X \times \mathbf{I}$ is called the cylinder on X, while the space $X^{\mathbf{I}}$ is called the cocylinder or path space on X.

When using an interval object to construct these two notions, the adjunction between products and exponentials ensure that the two notions of homotopy induced by the cylinder and cocylinder are the same. But for the purpose of abstraction, having one or the other structure without the adjunction or even the functoriality already suffices to capture the notion of homotopy.

Definition 4.6. Let X be an object of a category $\mathcal{C}$. A *cylinder object* $(CX, \top, \bot, p)$ for X is an object CX as well as maps $\top, \bot \colon X \to CX$ and $p : CX \to X$, such that $p\top = p\bot = id_X$. Dually, a *cocylinder object*[2] (PX, r, s, t) for X is an object PX as well as maps $r : X \to PX$ and $s, t : PX \to X$ such that $sr = tr = id_X$. ◊

Remark 4.7. In the classical case, the maps $p : X \times I \to X$ and $r : X \to X^I$ are homotopy equivalences. This is something we have not required, and amounts to asking for a contraction map, in the form of either $CX \to CCX$ or dually $PPX \to PX$. ⊚

When we construct a (co)cylinder object, the construction is often functorial, which is further captured by the following definitions.

Definition 4.8. A *functorial cylinder* $(C, \top, \bot, p)$ for a category $\mathcal{C}$ is an endofunctor $C : \mathcal{C} \to \mathcal{C}$ equipped with natural transformations $\top, \bot \colon id \Rightarrow C$ and $p : C \Rightarrow id$, such that $p\top = p\bot = id$. A map of functorial cylinders $\alpha : (C, \top, \bot, p) \to (D, \top, \bot, p)$ is a natural transformation $\alpha : C \Rightarrow D$ s.t. $\alpha\top = \top$, $\alpha\bot = \bot$ and $\alpha p = p$.

Functorial cocylinders (P, r, s, t) and their maps are defined dually. ◊

Of course, an exponentiable interval object I in $\mathcal{C}$ induces an adjoint functorial cylinder-cocylinder pair $(-) \times I \dashv (-)^I$.

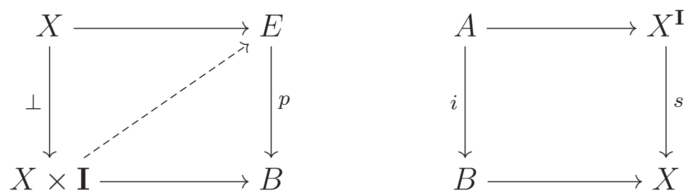
4.3 Model Structures

To take an even more abstract approach, one could do away with the (co)cylinders entirely and directly axiomatize the homotopy equivalences. In this case, we identify a class of maps in $\mathcal{C}$ called the *weak equivalences*, and try to axiomatize it. A major driving consideration is to axiomatize in such a way that we can recover a notion of homotopy from a given class of weak equivalences.

The solution, originally due to Quillen [Qui67], is to axiomatise two more classes of maps that play a role in the classical homotopy theory of topological spaces, and these are the *fibrations* and *cofibrations*. Classically, they are maps that "play nicely" with homotopy, in the following sense. Fibrations $p : E \to B$ allow homotopies onto B to be lifted to E, while cofibrations $i : A \to B$ allow homotopies onto A to be extended onto B.

[2]We will reserve the terminology of *path object* when we define path categories, where the s, t, r maps are required to satisfy additional conditions.

Definition 4.9. A map $p : E \to B$ of topological spaces is a Hurewicz fibration if for any square of the form on the left, a diagonal map exists making the triangles commute. Dually, a map $i : A \to B$ of topological spaces is a Hurewicz cofibration if for any square of the form on the right, a diagonal filler exists.



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The classical cocylinder (resp. cylinder) object can be presented as the following factorization of the diagonal (resp. codiagonal) into a homotopy equivalence followed by a fibration (resp. cofibration followed by a homotopy equivalence).

Now, the key property of fibrations and cofibrations is that we can factor any map into a homotopy equivalence (that is also a cofibration) followed by a fibration, or a cofibration followed by a homotopy equivalence (that is also a fibration). By axiomatising this factorisation property, we are essentially saying that factorisations of the (co)diagonal always exist, i.e. that (co)cylinder objects exist. From this, we can obtain a notion of homotopy. This exposition for considering model structures was inspired by [DS95], which we also refer to for proper introductory exposition, especially on recovering the (co)cylinders and homotopy.

Definition 4.10. A *model structure* on a category $\mathcal{C}$ with finite limits and colimits has the data of three distinguished classes of maps $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$

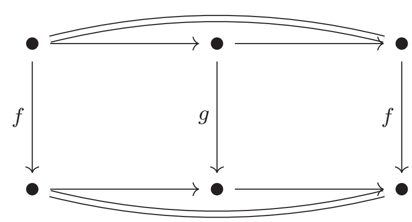
1. *weak equivalences* W ($\xrightarrow{\sim}$)
2. *fibrations* Fib ($\twoheadrightarrow$)
3. *cofibrations* Cof ($\rightarrowtail$)

We call $\mathsf{W} \cap \mathsf{Fib}$ the *acyclic fibrations* and $\mathsf{W} \cap \mathsf{Cof}$ the *acyclic cofibrations*. We further require the classes to satisfy the conditions

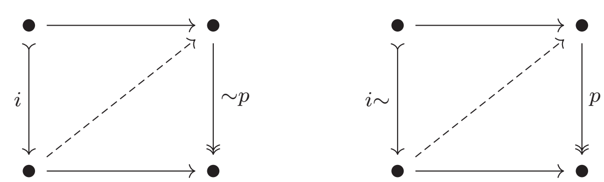
MC1 The three classes above are closed under composition and contains all identity maps.

MC2 If two of the three maps f, g, gf are weak equivalences then so is the third.

MC3 If f is a retract of g and g is a fibration/cofibration/weak equivalence, then so is f. The map f is a retract of g when there is a commutative diagram as follows.



MC4 Any cofibration i has the *left lifting property* with respect to acyclic fibrations p, and any fibration p has the *right lifting property* with respect to acyclic cofibrations i. That is to say, diagrams of the following form have diagonal fillers:



MC5 Every map factors as either a cofibration followed by an acyclic fibration or as an acyclic cofibration followed by a fibration.

When the map $X \to 1$ is a fibration, we say that X is *fibrant* and dually if the map $0 \to X$ is a cofibration, we say that X is *cofibrant*. $\Diamond$

Indeed, a model structure has been constructed on the category of topological spaces for which the weak equivalences are homotopy equivalences, due to Strøm. This is what we call the classical model structure on topological spaces.

Theorem 4.11 ([Str72, Theorem 3])**.** *There is a model structure on* Top *where*

$$\mathsf{Fib} := \{ \textit{ Hurewicz fibrations } \} \qquad \mathsf{Cof} := \{ \textit{ closed Hurewicz cofibrations } \}$$

$$\mathsf{W} := \{ \textit{ Homotopy equivalences } \}$$

The advantage of this abstract approach is that we can use it even to model notions of equivalence which is not known to be induced by a notion of homotopy. For example, Quillen's first example of a model structure on topological spaces [Qui67] actually has a weak equivalence $f : X \to Y$ as a map that induces an isomorphism on all homotopy groups of X and Y.

4.4 Path Categories

Both cofibrations and fibrations play a meaningful and foundational role in classical homotopy theory, hence their use in defining model categories. However, for homotopical interpretations of type theory, the fibrations take on a much more foundational role. A type-in-context $\Gamma \vdash A$ is interpreted as the prototypical fibration AKA projection maps $\Gamma.A \to \Gamma$. Following this emphasis on fibrations is also heightened emphasis on cocylinder objects over cylinder objects, since the identity types are interpreted using cocylinders. From a purely pragmatic standpoint, model categories are also less than ideal because of how difficult it is to obtain the factorization.

The natural thing to do then is to drop the class of cofibrations, since they are not as important for modelling type theory. This leads us to the notion of a path category [BM18].

Definition 4.12 ([BM18, Definition 2.2])**.** Let $\mathcal{C}$ be a category equipped with two classes of maps called fibrations and equivalences. We call a map an *acyclic fibration* or *trivial fibration* if it is both a fibration and an equivalence. Then $\mathcal{C}$ is a *path category* if the following axioms are satisfied.

PC1 Every object X has a cocylinder object (PX, r, s, t) such that r is an equivalence and $\langle s, t \rangle$ is a fibration. We call this a *path object* for X.

PC2 Fibrations are closed under composition, and pullback of fibrations exist and is still a fibration. Moreover, $\mathcal{C}$ has a terminal object and every map $X \to 1$ is a fibration.

PC3 The equivalences satisfy 2-out-of-6, i.e. if f, g, h are three composable maps and both gf and hg are equivalences, then so are f, g, h and hgf.

PC4 Every isomorphism is an acyclic fibration and acyclic fibrations are closed under pullbacks. Moreover, every acyclic fibration has a section.

$\diamondsuit$

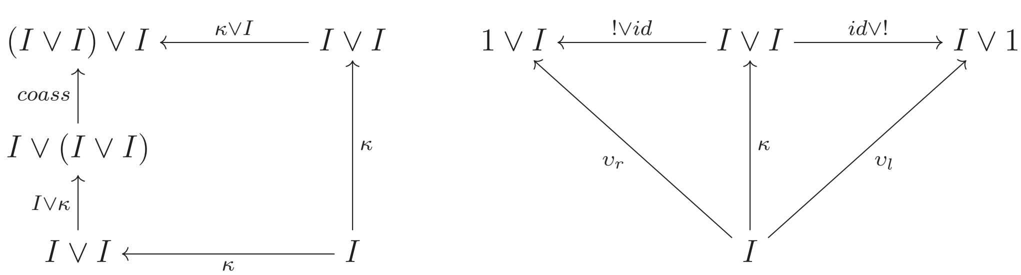
There are some additional changes relative to a model structure. For one, in a model structure not every object is required to be fibrant. Similarly, acyclic fibrations having sections is an implicit way of saying every object is cofibrant, but this is not generally true in a model structure. Finally The 2-out-of-6 requirement for equivalences is stronger than the 2-out-of-3 requirement, and in fact this condition is instrumental in proving that the equivalences are exactly the homotopy equivalences.

4.5 Path Categories from Interval Objects

If we have an interval object that is particularly well-behaved, then we can obtain a path category structure from its induced cocylinder object in a purely categorical fashion.

First, we require that the cut map be *strictly* co-associative and co-unital, which essentially allows us to manipulate paths in a very algebraic manner.

Definition 4.13. An interval object $(I, \perp, \top, \kappa)$ is *strict* if the following diagrams commute,



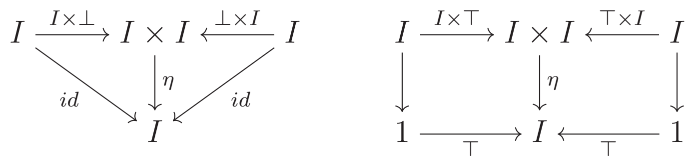
where *coass* is the induced map $[\vee_l \vee_l, [\vee_l \vee_r, \vee_r]]$ which simply redistributes the I-components. $\diamondsuit$

Example 4.14. The standard interval $\mathbf{I} = [0, 1]$ in Top is *not* strictly co-associative, since the two $\mathbf{I} \to (\mathbf{I} \vee \mathbf{I}) \vee \mathbf{I}$ induce two different subdivisions of $\mathbf{I}$ into three components:

Similarly, it is not strictly co-unital because $\kappa(! \vee id)$ contracts one half of the interval to an endpoint while stretching the other half, whereas υ_r is the identity map. However, **I** is co-associative and co-unital up to homotopy. ◊

Second, we require a triangle that represents a contraction of the interval to one of its endpoint.

Definition 4.15. A *contraction map* for an interval $(I, \bot, \top)$ is a map $\eta : I \times I \to I$ such that the following diagrams commute.



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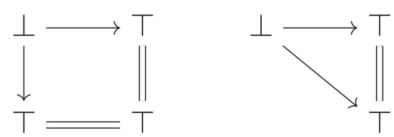
Remark 4.16. Pictorially, $I \times I$ is a square, and a contraction map produces the following lefthand square in I, which we can think of as a contraction of I to the constant path on its target point (righthand triangle, contracting downwards).



⊚

Notation 4.17. The structures on the interval induce corresponding maps on the cocylinder $X^{(-)}$, natural over X. We will use the following notation to denote these maps.

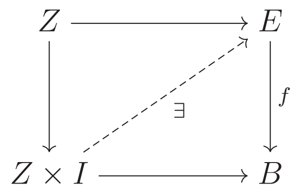
$$r : X \cong X^1 \to X^I \qquad s, t : X^I \to X^1 \cong X \qquad \tau : X^I \to X^I$$

$$\mu : X^I {}_t\!\times_s X^I \cong X^{I \vee I} \to X^I \qquad \eta : X^I \to X^{I \times I} \cong \left(X^I\right)^I$$

Strict interval objects $(I, \bot, \top, \tau, \kappa)$ exactly corresponds to the "object of co-morphisms" for internal co-groupoid objects in $\mathcal{C}$ whose "object of co-objects" is 1, as in [BG12, Definition 5.4.1]. As a consequence, the above maps make X^I into an internal groupoid of paths on X. ◊

In order to give a path category structure, we need a notion of fibration. This we can define in analogy with classical homotopy theory.

Definition 4.18. Let $(I, \bot, \top)$ be an interval object. A map $f : E \to B$ is an h_I-fibration if it has a lift for every square of the form



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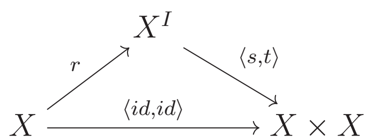
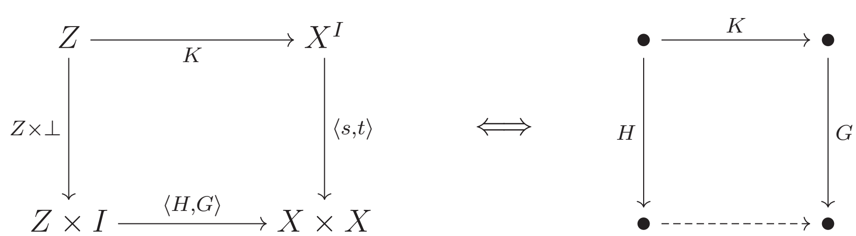
Proposition 4.19. *For a strict interval object* $(I, \bot, \top, \tau, \kappa, \eta)$ *and object* $X \in \mathcal{C}$ *there is a factorization*



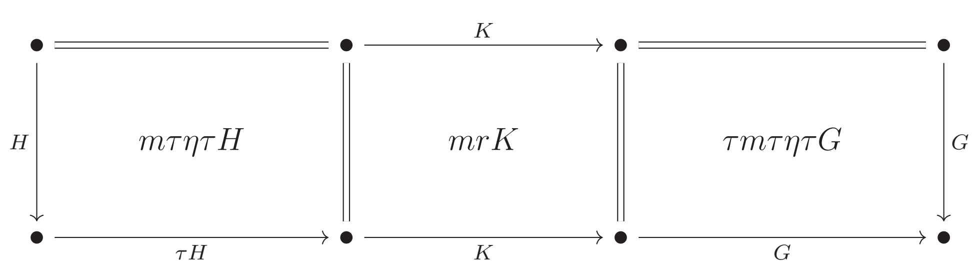
where r *is an* I*-homotopy equivalence and* $\langle s, t\rangle$ *is an* h_I*-fibration.*

Proof. First, note that the left adjoint of $sr = X \to X^I \to X^1$ is $X \times 1 \to X \times I \to X = \pi_X$, so $sr = id_X$, and similarly for tr. Now, r has homotopy inverse t, with the homotopy $rt \simeq id_{X^I}$ witnessed by η.

Next, to see that $\langle s, t\rangle$ is a h_I-fibration, observe that the following lefthand lifting square corresponds to the righthand shape, which we want to fill:



We can fill it by composing three squares in the following way:



The top face is equal to K by strict unitality. More formally, we can define the filler $F : Z \to (X^I)^I$ by

$$F = m\mu \left\langle \mu \left\langle m\tau\eta\tau H, mrK \right\rangle, \tau m\tau\eta\tau G \right\rangle$$

where $m : (X^I)^I \to (X^I)^I$ is the mirror map which mirrors a square, i.e. permutes the two Is in $(X^I)^I$. □

The previous proposition gives our path object. We now focus on the acyclic fibration axioms, which follows from a characterization of the acyclic h_I-fibrations in terms of fibred homotopies, defined as follows.

Definition 4.20. For an interval object $(I, \bot, \top)$, And two maps $f, g : E \to D$ in the slice category $\mathcal{C}/B$ from $p : E \to B$ to $q : D \to B$, a fibred I-homotopy $f \simeq_B g$ is a homotopy $H : f \simeq g$ such that $q^I H = rp$. The notion of fibred I-homotopy equivalence is defined analogously. ♢

Remark 4.21. For any interval object $(I, \bot, \top, \tau, \kappa)$, fibre homotopy is also a congruence relation, using the same operations induced by τ and κ. One simply needs to check that the fibre condition is preserved. ⊚

The following lemma shows that any homotopy equivalence between two h_I-fibrations is already automatically fibred.

Lemma 4.22 ([May99, Section 7.5])**.** *Let $(I, \bot, \top, \tau, \kappa, \eta)$ be an exponentiable interval object. Let there be a commutative diagram of the form*

$$\begin{array}{ccc} E & \xrightarrow[\sim]{f} & D \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

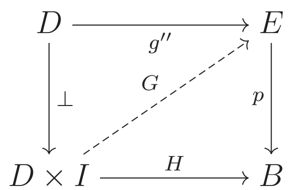
where f is an I-homotopy equivalence and p, q are h_I-fibrations. Then f is an I-homotopy equivalence fibered over B.

Proof. Let f have homotopy inverse g''. It suffices to find a one-sided fibered homotopy inverse g with $fg \simeq_B id_D$, since then

$$id_D \simeq g''f \implies g \simeq g''fg \simeq g''$$

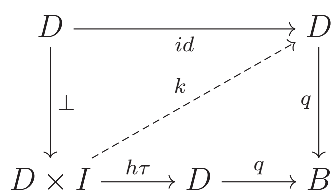
so g is a homotopy equivalence. Then by symmetry we can repeat the argument to find f' with $gf' \simeq_B id_E$, and therefore conclude $f' \simeq_B fgf' \simeq_B f$, meaning that $gf \simeq_B gf' \simeq_B id_E$.

Let us now construct g. We have a homotopy $H : pg'' \simeq q$, so first we replace g'' by a g' such that $pg' = q$. This is done by lifting

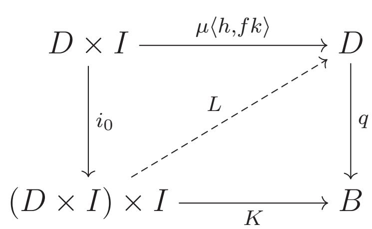


and defining $g' := G\top$. Now, if we can find a fibered right homotopy inverse e of $fg' : D \to D$ then we are done by taking $g := g'e$, since $fg = fg'e \simeq_B id_D$. The essential property of fg' for this construction is that $q(fg') = q$ and that there is a homotopy $h : fg' \simeq id_D$.

Renaming fg' to f for simplicity, we proceed with the construction of e. First, we take the lift



and define $e = k\top$. The key property of k is that $qfk = qk$ is equal to the inverse of qh. This is useful because we then have a homotopy $id_D \overset{h}{\simeq} f \overset{fk}{\simeq} fe$ such that the q-image of this homotopy is the homotopy qh followed by its inverse. This q-image can be deformed to the constant homotopy. Since q is a fibration, this allows us to find a homotopy $id_D \simeq fe$ which lives over the constant path, i.e. $id_D \simeq_B fe$ as desired. This is all just intuition, so now we carry out the proof formally. We perform the lift



where K is the composite of two higher homotopies as pictured on top, and the lift L as pictured below.

$$\begin{array}{ccccc}
q & \xrightarrow{\quad qh \quad} & qf = q & \xrightarrow{\quad qfk=qk=\tau qh \quad} & qfe = qe = qf = q \\
{\scriptstyle r}\big\downarrow & \tau\eta\tau qh & \big\downarrow{\scriptstyle \tau qh} & \eta\tau qh & \big\downarrow{\scriptstyle r} \\
q & \xrightarrow[\quad r \quad]{} & q & \xrightarrow[\quad r \quad]{} & q
\end{array}$$

$$\begin{array}{ccccc}
id_D & \xrightarrow{\;h\;} & f & \xrightarrow{\;fk\;} & fe \\
\vdots & & & & \vdots \\
\downarrow & & & & \downarrow \\
\bullet & & \dashrightarrow & & \bullet
\end{array}$$

Then the composite of the left face, bottom face followed by right face of L defines the desired homotopy $id_D \simeq_B fe$, which is a fibred homotopy because its q-image is the reflexive homotopy rq. □

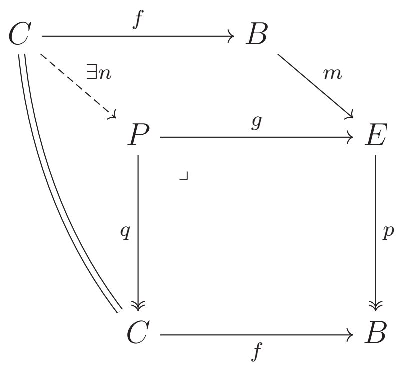
Corollary 4.23. *Every acyclic h_I-fibration $p : E \overset{\sim}{\twoheadrightarrow} B$ has a section that is a homotopy inverse of p fibered over B.*

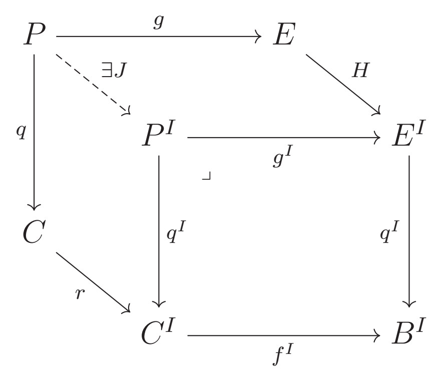
Proof. Take $f := p$, $q := id_B$ in the previous lemma. □

Theorem 4.24. *Given a strict interval object $(I, \bot, \top, \tau, \kappa, \eta)$ on a category $\mathcal{C}$ there is a path category structure on $\mathcal{C}$ where the fibrations are h_I-fibrations and equivalences are I-homotopy equivalences.*

Proof. Axiom (PC1) is obtained by considering X^I for each X. For axiom (PC2), closure under composition and pullbacks is easy to see for maps defined by RLP, and one can see that the h_I-fibration lifting condition trivially holds for $X \to 1$ by taking the reflexive homotopy. For axiom (PC3), The 2-out-of-6 property for I-homotopy equivalences follow purely formally from the fact that $\simeq$ is a congruence relation.

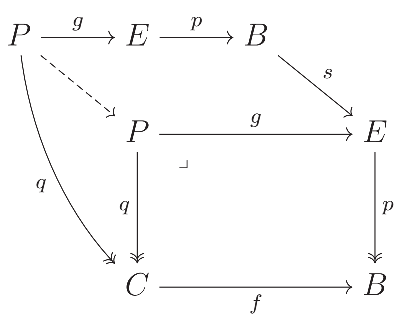
For axiom (PC4), it is also easy to see that every isomorphism is an I-homotopy equivalence and an h_I-fibration, and the previous lemma shows every acyclic fibration has a section. Finally, for closure under pullbacks, this is also a corollary of the previous lemma. That is, suppose we have a section $m : B \to E$ of an acyclic fibration $p : E \overset{\sim}{\twoheadrightarrow} B$ with a fibrewise homotopy $H : mp \simeq_B id_E$. Then on a pullback q of p, we can construct a section n and the homotopy inverse witness J as follows:

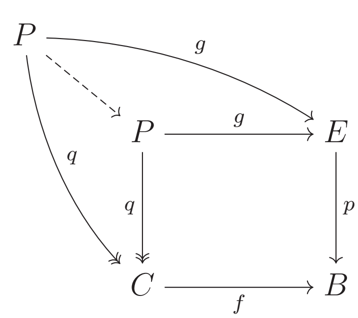




Note that the outer squares on the left and the right commute only because m is a section and because H is a fibrewise homotopy, respectively.

To see that $sJ = nq$ and $tJ = id_P$, we note that they respectively induce the same cones, which are





This shows that q is an I-homotopy equivalence. □

Chapter 5

Homotopy Theory in QCB

In the last chapter, we introduced the classical or Strøm model structure on Top. It is natural then to ask whether this model structure restricts to the sub-category of QCB spaces. As we will see, it does restrict to QCB but the since limits and colimits are generally different in QCB, the proof is less trivial than expected. In particular, while the cofibrations in QCB are precisely the closed Hurewicz cofibrations between QCB spaces in Top, the same cannot be said for the fibrations.

5.1 The Strøm Model Structure on QCB

The definition of homotopy and homotopy equivalence is the same for QCB spaces, since the QCB-product $X \times \mathbf{I}$ has the same topology as it would in Top (Theorem 3.29). We replicate the definitions of Hurewicz fibrations and cofibrations for QCB.

Definition 5.1. Let E, B, A, X, Y denote QCB spaces.

1. A *Hurewicz fibration* or *h-fibration* is a map $p : E \to B$ with RLP against endpoint inclusion maps $i_0 := X \times \perp : X \to X \times \mathbf{I}$ for any X.
2. A *Hurewicz cofibration* or *h-cofibration* is a map $j : A \to X$ with LLP against endpoint projection maps $p_0 := Y^{\perp} : Y^{\mathbf{I}} \to Y$ for any Y.
3. The *Strøm model structure* on QCB is the model structure where

$$\mathsf{Fib} := \{\, h\text{-fibrations} \,\} \qquad \mathsf{Cof} := \{\, \text{closed } h\text{-cofibrations} \,\}$$
$$\mathsf{W} := \{\, \text{homotopy equivalences} \,\}$$

$\Diamond$

Remark 5.2. A subtlety is that *a priori*, not all Hurewicz (co)fibrations in QCB are Hurewicz (co)fibrations of QCB spaces in Top. So let us denote the Hurewicz (co)fibrations in Top as h_{Top}-(co)fibrations. As we will soon see, the h-cofibrations and the h_{Top}-cofibrations do in fact coincide, but we don't know if this is true for the fibrations. $\circledcirc$

We of course have to prove that this definition satisfies the model structure axioms. In this chapter, categorical constructions are defined as in QCB, unless otherwise stated. Axiom MC1 follows by composing lifts, while axiom MC2 follows by definition of homotopy equivalences. Axiom MC3 also follows from purely formal considerations of lifting diagrams.

Before we embark on the proof of MC4 and MC5, we have the following results which allow us to import h_{Top}-fibrations when they're almost in QCB.

5.2 The Mapping Cylinder

The most important construction for proving axiom MC4 is the following mapping cylinder construction.

Definition 5.3. Let $f : X \to Y$ be a map. Then the *mapping cylinder*[1] is the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;i_0\;} & X \times \mathbf{I} \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \longrightarrow & Mf \end{array}$$

$\Diamond$

The mapping cylinder is useful in that the lifting property characterizing a fibration can be re-arranged into a universal lifting problem against the mapping cylinder. Since this is a purely categorical result, we have it for free.

Proposition 5.4. *A map $j : A \to X$ is an h-cofibration iff the lifting*

[1]more affectionately known as the top-hat space.

diagram

$$\begin{array}{ccc} A & \longrightarrow & Mj^{\mathbf{I}} \\ {\scriptstyle j}\downarrow & \nearrow & \downarrow{\scriptstyle p_0} \\ X & \longrightarrow & Mj \end{array}$$

where the horizontal maps are the pushout maps from the definition of Mj.

We next observe further results we can obtain for free, following from the fact that Mf is defined as in Top.

Proposition 5.5. *The mapping cylinder pushout $M_{\mathsf{Top}}f$ in* Top *is already T_0, so $Mf \cong M_{\mathsf{Top}}f$.*

Proof. Note that for any open set $U \in \tau_Y$, the open set

$$NU := (U \times 1) + (f^{-1}[U] \times \mathbf{I}) \in \tau_{(Y\times 1)+(X\times\mathbf{I})}$$

is closed under the equivalence relation

$$(y, \star) \sim (x, t) \overset{\Delta}{\iff} f(x) = y \text{ and } t = 0$$

defining the Top-pushout quotienting. Indeed, we have that

$$\begin{aligned} (f(x), \star) \in U \times 1 &\iff f(x) \in U \\ &\iff x \in f^{-1}[U] \iff (x, 0) \in f^{-1}[U] \times \mathbf{I} \end{aligned}$$

A particular consequence of this is that the quotient-image of this set in $N_{\mathsf{Top}}f$ is also open. Now consider the three possible scenarios where two points are not equal under the quotient.

1. ($(y, \star) \not\sim (y', \star)$) Then $y \neq y'$, so WLOG there is $U_y \in \tau_Y$ such that $y \in U_y \not\ni y'$. Then NU_y separates $(y, \star)$ and $(y', \star)$ (under the quotient).

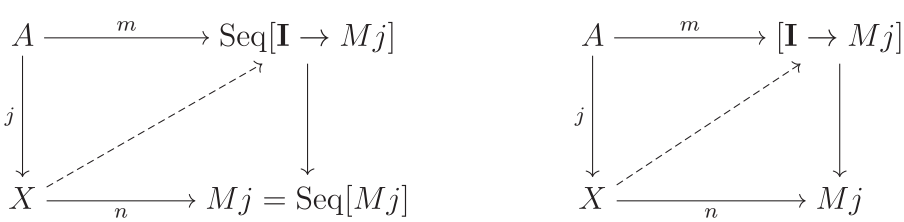
2. ($(y, \star) \not\sim (x, t)$) If $t = 0$, then $(x, 0) \sim (f(x), \star)$ and so the previous argument suffices to separate $(y, \star)$ and $(f(x), \star)$. If $t > 0$, then the open set $X \cup (t_0, t_1)$ for some $0 < t_0 < t < t_1$ separates the two points.

3. ($(x, t) \not\sim (x', t')$) Then $x \neq x'$ and either $f(x) \neq f(x')$ or one of t or t' is not 0. In any case, it will all eventually reduce to one of the previous cases.

□

Corollary 5.6. *Any h-cofibration is also an h_{Top}-cofibration.*

Proof. By the adjunction property of Seq, we have that the left square lifts iff the right square lifts.



□

Once we know this, we can import the standard results about h_{Top}-cofibrations without further effort.

Theorem 5.7 ([Str66, Theorem 1])**.** *If $j : A \to X$ is an h-cofibration then j is an embedding of topological spaces, i.e. j is a homeomorphism $A \cong j[A]$.*

This theorem shows that every cofibration is a "nice" inclusion of topological spaces. The following theorem characterizes "nice" as having some wiggle room for the construction of homotopies.

Definition 5.8. Let $A \subseteq X$ be a pair of topological spaces.

1. (X, A) is an *NDR (neighborhood deformation retract) pair* if there is a map $\varphi : X \to \mathbf{I}$ with $\varphi^{-1}(0) = A$ and a homotopy $h : X \times \mathbf{I} \to X$ such that
$$h(x,0) = x \qquad \forall x \in X$$
$$h(a,t) = a \qquad \forall a \in A, t \in \mathbf{I}$$
$$h(x,1) \in A \qquad \forall x \in \varphi^{-1}[0,1)$$
2. (X, A) is a *DR pair* if it is an NDR pair such that $\varphi^{-1}[0,1) = X$, in which case h is a deformation retraction of X onto A.

◊

Theorem 5.9 ([May99, Section 4.4])**.** *Let A be a closed subspace of a QCB space X, with the inclusion $i : A \hookrightarrow X$. The following are equivalent.*

1. (X, A) *is an NDR pair.*

2. $(X \times \mathbf{I}, Mi)$ *is a DR pair.*

3. Mi *is a retract of* $X \times \mathbf{I}$.

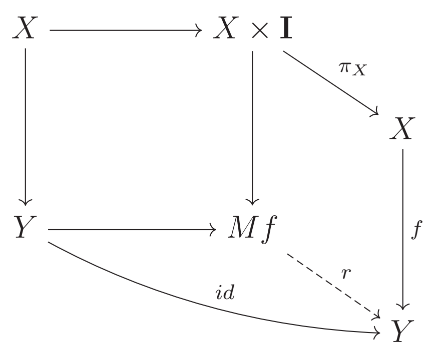
4. $A \to X$ *is an h-cofibration.*

Finally, we have a factorization result that will aid in proving MC5.

Proposition 5.10 ([May99, Section 4.3])**.** *Every map* $f : X \to Y$ *in* QCB *factors as*

$$X \overset{j}{\rightarrowtail} Mf \overset{\overset{r}{\sim}}{\to} Y$$

where $j(x) = (x, 1)$ *is an h-cofibration and* r *is the deformation retraction of the inclusion* $Y \to Mf$ *induced by*



5.3 The Lifting Axiom for QCB

With this analysis of cofibrations using the mapping cylinder, the lifting axiom MC4 follows. We first note that the definition of (acyclic) h-fibrations correspond precisely to the definition of acyclic $h_{\mathbf{I}}$-fibrations Definition 4.18, so we can import the result on fibre homotopy inverse.

since the proof of the following lemmas and proposition only uses the results of the previous analysis of cofibrations, the lifting properties of i and p against QCB spaces, plus some manual definition of homotopies to show that whenever p is acyclic, it has a section which is also a homotopy inverse.

We also have the following dual result, but this is a known result for h_{Top}-cofibrations.

Definition 5.11. Let $f, g : X \to Y$ be morphisms in the slice category $\mathsf{QCB}\backslash A$ from $i : A \to X$ to $j : A \to Y$. A homotopy $f \simeq^A g$ rel A is a homotopy $H : f \simeq g : X \times I \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times I & \xrightarrow{i \times I} & X \times I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle H} \\ A & \xrightarrow[j]{} & Y \end{array}$$

The notion of fibred I-homotopy equivalence is defined analogously. ◊

Lemma 5.12. *Every acyclic h-cofibration $i : A \overset{\sim}{\rightarrowtail} X$ has a retract that is a homotopy inverse of i rel A.*

Proposition 5.13 (MC4 for QCB [MP12, Proposition 17.1.4])**.** *For any lifting diagram*

$$\begin{array}{ccc} A & \xrightarrow{g} & E \\ {\scriptstyle i}\downarrow & \overset{\lambda}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & B \end{array}$$

in QCB*, if either i or p is a homotopy equivalence then this diagram has a lift.*

Proof. Suppose i is a homotopy equivalence. Then by the previous lemma, it has a retraction $r : X \to A$ and a homotopy $h : ir \simeq^A id_X$. Then the obvious thing to do is to construct the diagram

$$\begin{array}{ccccc} X & & \xrightarrow{gr} & & E \\ {\scriptstyle i_0}\downarrow & & \overset{v}{\nearrow} & & \downarrow{\scriptstyle p} \\ X \times \mathbf{I} & \xrightarrow[h]{} & X & \xrightarrow[f]{} & B \end{array}$$

and take $\lambda = vi_1$. However, this does not quite work because nothing about the above commutativities ensure that $\lambda i = g$. This would work if only we can take $\lambda(a) = v(a, 0)$ instead, while maintaining $\lambda(x) = v(x, 1)$ for $x \notin A$.

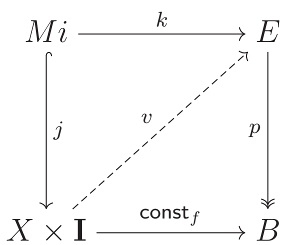
Luckily, (X, A) is an NDR-pair, so there is a separating function $\varphi : X \to \mathbf{I}$ s.t. $\varphi^{-1}(0) = A$. Then we can replace h by

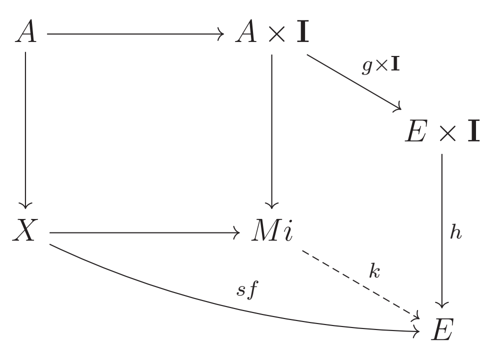
$$j(x,t) = \begin{cases} h(x, t/\varphi(x)) & \text{if } t < \varphi(x) \\ h(x,1) = x & \text{if } t \geq \varphi(x) \end{cases}$$

in the above lifting diagram, so now $pv = fj$. Since $\varphi(x)$ essentially measures the distance of x from A, j simply speeds up h so that it takes time $\varphi(x)$ to enter A. Then we can now take $\lambda(x) := v(x, \varphi(x))$. In particular, for elements $a \in A$ this distance is 0, so we can now ensure $\lambda(a) = v(a, 0) = gr(a) = g(a)$.

Suppose p is a homotopy equivalence. Then by the previous lemma, it has a section $s : B \to E$ with a fibrewise homotopy $h : sp \simeq_B id_E$. Then there is a lifting diagram

where





By the first half of this proof, the diagram has a lift v since i is a closed h-cofibration, so $(X \times \mathbf{I}, Mi)$ is a (N)DR pair and therefore both a homotopy equivalence and a closed h-cofibration. Taking $\lambda = vi_1$ yields the desired lift. $\square$

5.4 The Factorization Axiom for QCB

A construction of the $(\mathsf{Cof} \cap \mathsf{W}, \mathsf{Fib})$ factorization for Strøm's model structure on Top is given in [Rie14]. The construction relies on the Moore path space ΠY, which is the space of arbitrary (finite) length paths in Y. We adapt the construction to QCB. For this section, note that $\mathbb{R}^+$ is locally compact so Theorem 3.29 also applies to products with $\mathbb{R}$.

Definition 5.14. Let $\mathbb{R}^+$ denote the space of non-zero real numbers with the usual topology (which is QCB). We have the monoid structure of addition

$$0 : 1 \to \mathbb{R}^+ \qquad (- + -) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+.$$

From these, define the following maps by adjunction:

$$\frac{\pi_Y : Y \times \mathbb{R}^+ \to Y}{\mathsf{const} : Y \to Y^{\mathbb{R}^+}} \qquad \frac{Y^{(-+-)} : Y^{\mathbb{R}^+} \to Y^{\mathbb{R}^+ \times \mathbb{R}^+}}{\mathsf{shift} : Y^{\mathbb{R}^+} \times \mathbb{R}^+ \to Y^{\mathbb{R}^+}}$$

$\Diamond$

Definition 5.15. Let Y be a QCB space. The *Moore path space* ΠY is defined as the pullback

$$\begin{array}{ccc} \Pi Y & \xrightarrow{\langle p, \mathsf{length} \rangle} & Y^{\mathbb{R}^+} \times \mathbb{R}^+ \\ {\scriptstyle p_{\mathsf{end}}} \downarrow & \lrcorner & \downarrow {\scriptstyle \mathsf{shift}} \\ Y & \xrightarrow[\mathsf{const}]{} & Y^{\mathbb{R}^+} \end{array}$$

1. The reflexivity map $r := \langle id_Y, \langle \mathsf{const}, 0 \rangle \rangle : Y \to \Pi Y$ sends a point y to the constant path at y of length 0.
2. Let the Moore evaluation map be

$$\tilde{\text{ev}} : \Pi Y \times \mathbb{R}^+ \overset{p \times \mathbb{R}^+}{\to} Y^{\mathbb{R}^+} \to \mathbb{R}^+ \overset{\text{ev}}{\to} Y$$

3. Then the map $p_t := \Pi Y \overset{\langle id, t \rangle}{\to} \Pi Y \times \mathbb{R}+ \overset{\tilde{\text{ev}}}{\to} Y$ sends a Moore path to its value at time t, for any $t \in \mathbb{R}^+$.

$\Diamond$

Now, we define truncation and concatenation operations on Moore paths. Unfortunately, these cannot be defined purely categorically, at least not easily. Regardless, they are continuous, both in Top and QCB.

Lemma 5.16. *There are truncation and concatenation maps defined in the obvious way:*

$$\mathsf{trunc} : [\mathbb{R}^+ \to Y] \times \mathbb{R}^+ \to \Pi Y \qquad \mathsf{concat} : \Pi Y_{p_{\mathsf{end}}} \times_{p_0} \Pi Y \to \Pi Y$$

Proof. By various universal properties, these reduce to proving the maps

$$\mathsf{trunc} : Y^{\mathbb{R}^+} \times \mathbb{R}^+ \times \mathbb{R}^+ \to Y : (p, t, t') \mapsto \begin{cases} p(t') & t' \leq t \\ p(t) & t' > t \end{cases}$$

$$\mathsf{concat} : (\Pi Y_{p_{\mathsf{end}}} \times_{p_0} Y^{\mathbb{R}^+}) \times \mathbb{R}^+ \to Y : (((p, t), q), t') \mapsto \begin{cases} p(t') & t' \leq t \\ q(t' - t) & t' > t \end{cases}$$

are continuous. To prove their continuity in Top, we have to use the local compactness of $\mathbb{R}^+$ in order to find compact sets that map into any open set $V \in \tau_Y$, allowing us to construct open sets in the compact-open topology on $Y^{\mathbb{R}^+}$. The continuity in QCB follows from the fact that the domain spaces of trunc and concat have a finer topology in QCB than in Top, while the codomain space Y has the same topology. Hence, continuity in Top implies continuity in QCB. □

Using the Moore path space, we can define the Moore version of the mapping cocylinder, which is the space of Moore paths $f(x) \rightsquigarrow y$ in Y.

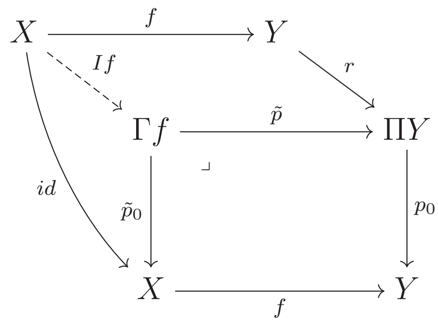
Definition 5.17. For any map $f : X \to Y$, the *Moore mapping cocylinder* is defined as the pullback

$$\begin{array}{ccc} \Gamma f & \xrightarrow{\tilde{p}} & \Pi Y \\ {\scriptstyle \tilde{p}_0}\downarrow & \lrcorner & \downarrow{\scriptstyle p_0} \\ X & \xrightarrow[f]{} & Y \end{array}$$

◊

Using this, we can factor $f : X \to Y$ as a map If that sends $x \in X$ to the zero-length constant path on $f(x)$ in Γf, and a map Mf that sends a Moore path $f(x) \to y$ to the endpoint y.

Definition 5.18. Let QCB^2 denote the arrow category of QCB. Define two functors $I, M : \mathsf{QCB}^2 \to \mathsf{QCB}^2$ by



$$Mf := \Gamma f \xrightarrow{\tilde{p}} \Pi Y \xrightarrow{p_{\mathsf{end}}} Y$$

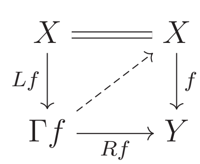
for any map $f : X \to Y$. ◊

We now have to show that Mf is a fibration, and that If is an acyclic cofibration. The crux of the argument lies in observing that

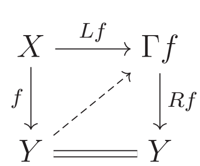
1. I, M are endofunctors with a counit and unit respectively.
2. Mf is a (free) M-algebra, and M-algebras correspond to h-fibrations.
3. If is a closed L-coalgebra, and any L-coalgebra lifts against any M-algebras AKA h-fibrations. Hence, any closed L-coalgebra is an acyclic h-cofibration.

However, we are able to import these results because their proofs all use maps that can be defined purely categorically from the maps that we have previously defined.

Lemma 5.19 ([Rie14, Lemma 3.7])**.** *The M-algebra structures for $f \in \mathsf{QCB}^2$ correspond to lifts*



Dually, the I-algebra structures for $f \in \mathsf{QCB}^2$ correspond to lifts



Proposition 5.20 ([May75, Proposition 3.4])**.** *A map $p : E \to B$ is an h-fibration iff it admits the structure of an M-algebra.*

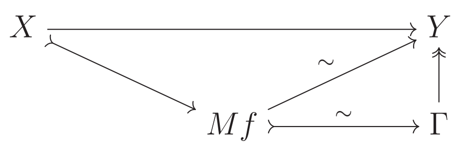
Lemma 5.21 ([Rie14, Lemma 3.11])**.** *M is a monad, and so Mf is a (free) algebra for M.*

Theorem 5.22 ([Rie14, Corollary 3.12])**.** *Let $f : X \to Y$. Then If is a closed acyclic h-cofibration, and Mf is a fibration.*

Since we already have a $(\mathsf{Cof}, \mathsf{W})$ factorization, we can obtain a $(\mathsf{Cof}, \mathsf{W} \cap \mathsf{Fib})$ factorization in the standard way.

Corollary 5.23 (MC5 for QCB)**.** *Any map f in QCB factors as an acyclic h-cofibration followed by a fibration, or as an h-cofibration followed by an acyclic h-fibration.*

Proof. For the second factorization, perform the following factorizations



and observe that the rightmost fibration is also a homotopy equivalence by two-out-of-three. □

This concludes the proof of the classical model structure on QCB.

Chapter 6

Homotopy Theory in Equ

The standard unit interval $\mathbf{I} = [0, 1]$ can be considered as an equilogical space with a discrete equivalence relation, which in principle should allow the development of a "classical" homotopy theory in Equ. However, a moment's thought reveals a major issue: because morphisms in Equ are equivalence classes, two paths agree on endpoints only up to equivalence. However, in general there is no way to stitch together two such paths into a single continuous map from $\mathbf{I}$.

However, consider what is meant by a "classical" homotopy theory in Equ. As we explained in the introduction, we wish for our homotopy theory in Equ to represent or implement the classical homotopy theory in QCB. To be really liberal about this interpretation, any notion of path/homotopy in Equ that collapses to the usual one in QCB under quotienting will do. One solution then is to consider a path in X not just as a map $[0, 1] \to X$ (henceforth called a naive path), but more generally as n-chains of naive paths, i.e. a path is a map $\mathbf{I}^n \to X$ where $\mathbf{I}^n := \mathbf{I} \vee \mathbf{I} \ldots \vee \mathbf{I}$. Then composition of an n-path with an m-path produces an $(n + m)$-path.

The notion of homotopy can be generalized in a similar fashion, but there is an ambiguity. If we think of a homotopy $H : f \simeq g : X \to Y$ as a family of paths in Y indexed by X (i.e. the cocylindrical perspective), then $H(x)$ can in principle have a different length from $H(x')$ provided $x \neq x'$. That is to say, we have a *length-local* notion of homotopy. On the other hand, if we think of a homotopy as an I-shaped movement of X-shapes in Y (i.e. the cylindrical perspective), then I here should be some $\mathbf{I}^n$ which fixes the length for all paths in the homotopy to n. That is, a *length-global* notion of homotopy.

In this chapter, we explore the plausibility of developing homotopy theory in Equ along these lines of thought. Rather than tackling the problem directly[1], we reconceptualize the problem as that of *amalgamating* two homotopy theories.

The key observation is that Equ is already a homotopy category with respect to the interval object

$$\mathcal{I} := (\{ 0, 1 \}, 0 \sim 1).$$

In section 6.1, we introduce the category Eql whose objects are equilogical spaces, but morphisms are actually equivariant maps, and not equivalence classes. We show that the path category structure on Eql induced by $\mathcal{I}$ has homotopy category Equ. This observation was inspired by [Ber20], where the effective topos Eff is shown to be a homotopy category. In section 6.2, we generalize this result on Eff to show $\mathsf{RT}(\mathbb{P})$ is already a homotopy category of a category $\mathbb{RT}(\mathbb{P})$, and relate the homotopy functor $\mathsf{Eql} \to \mathsf{Equ}$ as a kind of pullback of $\mathbb{RT}(\mathbb{P}) \to \mathsf{RT}(\mathbb{P})$ along the inclusion of Equ as a full subcategory of $\mathsf{RT}(\mathbb{P})$.

In section 6.3, we introduce a second path category structure on Eql induced by the usual interval $[0, 1]$. Since morphisms in Eql are up to equality, the notion of paths and homotopy can be taken as the naive notion, and everything works out as expected. Then the homotopy theory for Equ sketched above can be thought of as a manifestation of some homotopy theory in Eql which amalgamates the two homotopy theories we have constructed on Eql, which we describe in detail above. Taking amalgamation to mean the pushout of homotopy categories, the notion of homotopy that should be modelled ends up being the length-global one.

In section 6.4, we present some negative results on the impossibility or triviality of amalgamating length-global homotopy theories in Eql. In particular, we show that there is no model category on Equ for which the length-global homotopy equivalences are the weak equivalences. We also show that in order to make the jointed cocylinder object $X^{\mathcal{I} \vee [0,1]}$ a path object in a path category, we are forced to restrict to a vanishingly tiny subcategory of Eql.

These results would suggest at least that length-global homotopy is not an appropriate notion of homotopy, and that our conception of amalgamation as pushout of homotopy categories is mistaken. However, it also seems

[1] which I initially attempted to great pain...

impossible to construct a well-behaved path object PY which induces the notion of local-homotopy, as documented in Appendix B. The impossibility boils down to a *seemingly* technical conflict between equivariance and continuity, but actually it may yet be possible to give a conceptual explanation: the path object exists in $\mathbb{RT}(\mathbb{P})$ or $\mathsf{RT}(\mathbb{P})$, but its construction requires the . We briefly explore this in section 6.5.

6.1 Equ is already a Homotopy Category

The pushout $[0,1] \vee [0,1]$ is computed as the equilogical space

$$([0,1] + [0,1], \upsilon_l(1) \sim \upsilon_r(0))$$

with two disconnected components in the underlying space, which prevents us from defining a continuous cut map $[0,1] \to [0,1] \vee [0,1]$. So, $[0,1]$ cannot cut because colimits are computed by adding additional equivalence relations instead of by gluing. If we consider the equivalence as a kind of path, then colimits in Equ actually resemble some form of homotopy colimit, which suggests that Equ is already a homotopy category. Another feature of Equ clues us in: the morphisms are already equivalence classes of maps, which also tells us what category Equ is the homotopy category of - it is the category of equilogical spaces where maps are simply equivariant continuous functions (without taking equivalence classes).

Definition 6.1. Let Eql denote the category where the objects are equilogical spaces, and the morphisms are equivariant continuous functions. We have a forgetful functor $U : \mathsf{Eql} \to \omega T_0$. $\Diamond$

We can easily express the required notion of homotopy via an interval object, albeit a rather degenerate one.

Definition 6.2. Let the *computational interval* $(\mathcal{I}, \bot, \top)$ in Eql be defined by the discrete two-element space $1+1$, equipped with the maximal equivalence relation. Let $\bot$ and $\top$ denote distinct points of $\mathcal{I}$. $\Diamond$

Remark 6.3. This choice of topology on the two element space is not the only one. We can also consider the Sierspinski topology, in which case we obtain some sort of directed homotopy. ⊚

Categorical Constructions in Eql

Before we further explore the homotopy theory of $\mathcal{I}$, we show that categorical constructions in Eql are computed as in ωT_0.

Lemma 6.4. *Let $D : J \to \mathsf{Eql}$ be a diagram. If UD has a (co)limit, then so does D.*

Proof. Suppose UD has a limiting cone $(X, \gamma_j : X \to |D_j|)$. Then the maximal equivalence relation on X such that each γ_j is equivariant is defined by

$$x \sim_X x' \overset{\Delta}{\iff} \forall j \in J. \gamma_j(x) \sim \gamma_j(x').$$

Moreover, any cone of D already has a unique induced continuous map into X, so one simply checks that the induced map becomes equivariant with respect to $(X, \sim_X)$.

Suppose UD has a limiting cocone $(X, \gamma_j : |D_j| \to X)$. Then the equivalence relation generated by

$$x \sim x' \overset{\Delta}{\iff} \exists j \in J, d \in \gamma_j^{-1}(x), d' \in \gamma_j^{-1}(x'). d \sim d'$$

is the minimal equivalence relation such that each γ_j is equivariant. There is already a unique induced continuous map from X to any cocone of D, one then simply checks that the induced map is equivariant. □

Lemma 6.5. *Let X and Y be equilogical spaces, and let (W, ev) be an exponential object $|Y|^{|X|}$ in ωT_0. Then the exponential in Eql is the subspace*

$$|V| := \{ w \in W \mid \forall x \sim_X x'. \mathrm{ev}(w, x) \sim_Y \mathrm{ev}(w, x') \}$$

equipped with the relation $w \sim_V w' \overset{\Delta}{\iff} \forall x. \mathrm{ev}(w, x) \sim_Y \mathrm{ev}(w', x)$, and the evaluation map is the restriction to $|V|$ of the ev *for W.*

Proof. By construction, ev restricts to an equivariant map on V. Then given a diagram in Eql on the left, we can induce the corresponding diagram in ωT_0 on the right, and use the universal property of W to uniquely induce $\tilde{f}$.

$$\begin{array}{ccc} X \times V & \xrightarrow{\mathrm{ev}} & Y \\ \uparrow {\scriptstyle X \times (\exists ! \tilde{f})} & \nearrow_{f} & \\ X \times Z & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} |X| \times W & \xrightarrow{\mathrm{ev}} & |Y| \\ \uparrow {\scriptstyle |X| \times (\exists ! \tilde{f})} & \nearrow_{f} & \\ |X| \times |Z| & & \end{array}$$

It is then easy to see that $\tilde{f}$ is an equivariant map $Z \to V$, satisfying the commutativity for the left diagram. Furthermore, uniqueness in the right diagram implies uniqueness in the left. □

Remark 6.6. In Equ, this result holds for any *weak* exponential W, not just exponentials. This makes sense since the exponential in Equ is really some kind of homotopy exponential. ◎

Computational Path Category Structure on Eql

It is clear that the axioms of the flip, cut and contraction map uniquely determine these maps on $\mathcal{I}$, since the axioms specify how the maps should behave on endpoints but the data of $\mathcal{I}$ is composed of only the two endpoints. It is then also clear that $\mathcal{I}$ is a strict interval.

We next need to show that $\mathcal{I}$ is exponentiable, but this is clearly true since the exponential of underlying spaces $|X|^{|\mathcal{I}|}$ can be taken simply as $|X| \times |X|$, which is ωT_0. Then, we can observe that the exponent $X^{\mathcal{I}}$ can be simply given as the sub-equilogical space of $|X| \times |X|$ containing the elements (x, x') satisfying $x \sim_X x'$.

Corollary 6.7. *$\mathcal{I}$ is a strict, exponentiable interval object in* Eql*, and therefore induce a path category on* Eql*. Moreover,* Equ *is the homotopy category of* Eql *with respect to this path category structure.*

A $\mathcal{I}$-homotopy only has endpoints, with no real data witnessing the homotopy. That is, if two maps are homotopic, then there is a unique homotopy witnessing this[2].

6.2 The Realizability Topos is already a Homotopy Category

This observation that Equ is already a homotopy category was inspired by [Ber20], where it was shown that the effective topos (i.e. the realizability topos over Kleene's Turing Machine model of computation) is equivalent to the homotopy category for a path category. The developments in [Ber20] make no essential use of the underlying model of computation, so we can

[2]In the HoTT terminology, our notion of homotopy is -1-truncated.

easily adapt the construction to $\mathbb{P}$. We show how our analysis relates to this construction of $\mathsf{RT}(\mathbb{P})$ as a homotopy category.

Definition 6.8. An object of $\mathbb{RT}(\mathbb{P})$ is a tuple $(A, \alpha, A(-,-), \mathbf{i}, \mathbf{s}, \mathbf{t})$ where

1. A is a set of 0-cells, with $\alpha : A \to \mathbb{P}$ assigning to each $a \in A$ a realizer α_a.
2. For each $a, a' \in A$, a subspace $A(a, a') \subseteq \mathbb{P}$ of 1-cells. Furthermore, the distinguished continuous maps $\mathbf{i}, \mathbf{s}, \mathbf{t} : \mathbb{P} \to \mathbb{P}$ compute identity, symmetry and transitivity for 1-cells, in the sense that for all $a, a', a'' \in A$ and for all $\pi \in A(a, a'), \pi' \in A(a', a'')$,

$$\mathbf{i}\alpha_a \in A(a,a) \qquad \mathbf{s}\pi \in A(a',a) \qquad \mathbf{t}\left\langle\langle \pi, \pi' \rangle\right\rangle \in A(a, a'')$$

We will leave the distinguished maps $\mathbf{i}, \mathbf{s}, \mathbf{t}$ implicit in notation, unless they need to be mentioned. ♢

Remark 6.9. The objects of $\mathbb{RT}(\mathbb{P})$ are to be thought of as a kind of *locally codiscrete* 2*-groupoid*. The 2-cells are implicit because codiscreteness means that between any two 1-cells $\pi, \pi' \in A(a, a')$, there is exactly one 2-cell, so there is no need to mention it. From this intuition, one can see that all the expected identity, inverse and composition axioms for a 2-groupoid are witnessed by a unique 2-cell, so there is no need to enforce them explicitly either. We will call this category the homotopy realizability topos because its homotopy category will be the topos $\mathsf{RT}(\mathbb{P})$, so it is a "homotopy topos". ⊚

Definition 6.10. A morphism $f : (B, \beta) \to (A, \alpha)$ in $\mathbb{RT}(\mathbb{P})$ is a function $f : B \to A$ such that there exist continuous maps $f_0, f_1 : \mathbb{P} \to \mathbb{P}$ with the property that for all $b, b' \in B$,

$$f_0\beta_b = \alpha_{f(b)} \qquad \forall \pi \in B(b, b').f_1 \left\langle\left\langle\langle\langle \beta_b, \beta_{b'} \rangle\rangle, \pi\right\rangle\right\rangle \in A(fb, fb').$$

We will denote by $f_{(b,b')} : B(b, b') \to A(fb, fb')$ the function

$$\pi \mapsto f_1 \left\langle\left\langle\langle\langle \beta_b, \beta_{b'} \rangle\rangle, \pi\right\rangle\right\rangle,$$

which will be part of the data of the morphism f (but not f_1 itself). ♢

Definition 6.11 ([Ber20, Definitions 3.3 & 3.4])**.** Two maps $f, g : (B, \beta) \to (A, \alpha)$ are $\mathbb{P}$*-homotopic* if there is a continuous map $h : \mathbb{P} \to \mathbb{P}$ such that for every $b \in B$, $h\beta_b \in A(fb, gb)$. Such a realizer will be called a *coded* $\mathbb{P}$*-homotopy*. This extends to a notion of $\mathbb{P}$-homotopy equivalence on $\mathbb{RT}(\mathbb{P})$.

A map $p : (E, \epsilon) \to (B, \beta)$ is a $\mathbb{P}$*-fibration* if there are continuous maps $\varphi_0, \varphi_1 : \mathbb{P} \to \mathbb{P}$ such that

- for any e, b and $\pi \in B(p(e), b)$,

$$\exists e' \in E, \rho \in E(e, e'). \quad \varphi_0 \langle\langle\langle\langle \epsilon_e, \beta_b \rangle\rangle, \pi\rangle\rangle = \langle\langle \epsilon_{e'}, \rho\rangle\rangle \text{ and } p(e') = b \text{ and } p(\rho) = \pi.$$

- for any $e, e' \in E$, $\rho \in E(e, e')$ and $\pi \in B(pe, pe')$,

$$\varphi_1 \langle\langle\langle\langle e, e' \rangle\rangle, \langle\langle \rho, \pi\rangle\rangle\rangle\rangle \in E(e, e') \text{ and } p\varphi_1 \langle\langle\langle\langle e, e' \rangle\rangle, \langle\langle \rho, \pi\rangle\rangle\rangle\rangle = \pi.$$

◊

Remark 6.12. With the 2-groupoidal intuition, φ_0 computes lifts of 1-cells while φ_1 computes lifts of 2-cells. ⊚

The proof that these two classes of maps form a path category makes use of only features that exist in any PCA such as pair-encodings, so the following result carries over.

Theorem 6.13 ([Ber20, Theorem 3.6])**.** $\mathbb{RT}(\mathbb{P})$ *is a path category with* $\mathbb{P}$*-homotopy equivalences and* $\mathbb{P}$*-fibrations.*

We now give the proof sketch that the homotopy category of $\mathbb{RT}(\mathbb{P})$ is indeed the realizability topos. For this we will give a partial definition of objects and morphisms in the realizability topos.

Definition 6.14. An object of $\mathsf{RT}(\mathbb{P})$ is a pair $(A, =_A)$ such that A is a set and $=_A: A \times A \to \mathcal{P}(\mathbb{P})$ is a function satisfying certain conditions that make $=_A$ a sort of partial $\mathcal{P}(\mathbb{P})$-valued equivalence relation.

A morphism $[F] : (B, =_B) \to (A, =_A)$ is represented by a $\mathcal{P}(\mathbb{P})$-valued relation $F : B \times A \to \mathcal{P}(\mathbb{P})$ satisfying certain internal functionality and equivariance conditions. Two such relations F, G represent the same morphism if there are realizers $\varphi, \psi : \mathbb{P} \to \mathbb{P}$ such that for all $(b, a) \in B \times A$ and $p \in \mathbb{P}$,

$$p \in F(b, a) \implies \varphi(p) \in G(b, a) \text{ and } p \in G(b, a) \implies \psi(p) \in F(b, a).$$

◊

Proposition 6.15 ([Ber20, Proposition 3.7])**.** *The homotopy category of* $\mathbb{RT}(\mathbb{P})$ *is equivalent to* $\mathsf{RT}(\mathbb{P})$*.*

Proof. Define the functor $P : \mathbb{RT}(\mathbb{P}) \to \mathsf{RT}(\mathbb{P})$ sending (A, α) to $(A, =_A)$ with

$$[a =_A a'] = \{ \langle\langle\langle\langle \alpha_a, \alpha_{a'} \rangle\rangle, \pi\rangle\rangle \mid \pi \in A(a, a') \}$$

and sending a map $f : (B, \beta) \to (A, \alpha)$ to the morphism represented by

$$F(b, a) = \{ \langle\langle\langle\langle \beta_b, \alpha_a \rangle\rangle, \pi \rangle\rangle \mid \pi \in A(fb, a) \}$$

Then Pf and Pg are represented by the same relation precisely when f and g are $\mathbb{P}$-homotopic, and moreover P is full (by using some axiom of choice or by using a suitably constructive definition of morphisms in $\mathsf{RT}(\mathbb{P})$).

Now, it suffices to show that P is essentially surjective. Given an object $(A, =_A)$ of $\mathsf{RT}(\mathbb{P})$, construct the following object $(\tilde{A}, \alpha)$ in $\mathbb{RT}(\mathbb{P})$:

$$\begin{aligned} \tilde{A} &:= \{ (a, p) \mid a \in A, p \in [a =_A a] \} \\ \alpha(a, p) &:= p \\ \tilde{A}((a, p), (a', p')) &:= [a =_A a'] \end{aligned}$$

Then one can show that $P(A', \alpha) \cong (A, =_A)$. □

Now, an equilogical space $(X, \sim)$ with a specified basis enumeration B induces an embedding $e_B : X \hookrightarrow \mathbb{P}$, with which it can be viewed as an object $(X/_\sim, =_X)$ of $\mathsf{RT}(\mathbb{P})$ with

$$[[x] =_X [x']] := \{ e_B(x'') \mid x'' \in [x] \text{ and } x'' \in [x'] \}$$

Next, an Eql-morphism $f : (X, \sim) \to (Y, \sim)$ in Eql induces a relation

$$\begin{aligned} F([x], [y]) &:= \{ \langle\langle x', y' \rangle\rangle \mid x \in [[x] =_X [x]] \text{ and } y' \in [[f(x)] =_Y [y]] \} \\ &= \begin{cases} \{ \langle\langle e_{B_X}(x'), e_{B_Y}(y') \rangle\rangle \mid x' \in [x], y' \in [y] \} & [y] = [f(x)] \\ \emptyset & [y] \neq [f(x)] \end{cases} \end{aligned}$$

If $f \sim f'$ then the induced F and F' represent the same morphism in $\mathsf{RT}(\mathbb{P})$, and so we have:

Definition 6.16 ([Oos08])**.** The above definitions yield a functor $i : \mathsf{Equ} \to \mathsf{RT}(\mathbb{P})$. ♢

Combining this definition with the construction for essential surjectivity above, we obtain from $(X, \sim)$ the object (X, ξ) of $\mathbb{RT}(\mathbb{P})$ given by $\xi_x = e_B(x)$ and

$$X(x, x') := \{ e_B(x'') \mid x \sim x'' \sim x' \}$$

To define the functor $\mathsf{Eql} \to \mathbb{RT}(\mathbb{P})$, we can simply take the map f itself. Then the witness f_0 can be obtained by applying the extension theorem of $\mathbb{P}$ to f, and f_1 can be constructed from f_0, as we show in the following theorem.

Theorem 6.17. *The above definition yields a fully faithful functor $\tilde{i} : \mathsf{Eql} \to \mathbb{RT}(\mathbb{P})$ such that the following diagram commutes (up to isomorphism).*

$$\begin{array}{ccc} \mathsf{Eql} & \xrightarrow{\text{Ho}} & \mathsf{Equ} \\ \tilde{i} \big\downarrow & & \big\downarrow i \\ \mathbb{RT}(\mathbb{P}) & \xrightarrow[P]{} & \mathsf{RT}(\mathbb{P}) \end{array}$$

Proof. First, we show that $\tilde{i}$ is well-defined. We first have to ensure we can construct the $\mathbf{i}, \mathbf{s}, \mathbf{t}$ data for $\tilde{i}(X, \sim) := (X, \xi)$. We can take $\mathbf{i}$ to be the identity map $id : \mathbb{P} \to \mathbb{P}$, since it would send $\xi_x = e_B(x)$ to $e_B(x) \in X(x, x)$. Similarly, $\mathbf{s}$ can be the identity since $X(x, x') = X(x', x)$. Finally, for $\mathbf{t}$ it suffices to take either the left or right projection (doesn't matter which). Next, we have to show that given a map $f : (X, \sim) \to (Y, \sim)$, the function $\tilde{i}(f) := f : \tilde{i}(X, \sim) = (X, \xi) \to (Y, \upsilon) = \tilde{i}(Y, \sim)$ has the correct realizers f_0 and f_1. We can induce f_0 by applying the extension theorem of $\mathbb{P}$ to f, which gives it the property that $f_0 e_{B_X}(x) = e_{B_Y}(f(x))$. We can then take $f_1 := f_0 \circ p_2$ where p_2 projects to the second component of an encoded pair. This has the property that

$$f_1 \langle\langle\langle\langle \xi_x, \xi_{x'} \rangle\rangle , e_{B_X}(x'') \rangle\rangle = f_0 e_{B_X}(x'') = e_{B_Y}(fx'') \in Y(fx, fx')$$

for any $e_{B_X}(x'') \in X(x, x')$. This shows that f_0 and f_1 are appropriate realizers for f, noting that the action on 1-cells induced by f_1 is independent of the choice of f_0.

This is clearly a faithful functor, and it is also easy to see that it is full since if f is any map $\tilde{i}(X, \sim) \to \tilde{i}(Y, \sim)$, then $x \sim x'$ implies $X(x, x')$ is non-empty, so the existence of the realizer f_1 implies $Y(fx, fx')$ is also non-empty and therefore $fx \sim fx'$. That is, any map f between objects in the image of $\tilde{i}$ is an equivariant map on the original equilogical spaces.

Next, the commutativity of the diagram on objects amounts to showing that $i(X, \sim) = (X/_{\sim}, =_{X/_{\sim}}) \cong (X, =_X) = P(\tilde{i}(X, \sim))$. But this isomorphism $[F]$ and its inverse can be represented by

$$F(x, [x']) = F^{-1}([x'], x) := [[x] =_{X/_{\sim}} [x']].$$

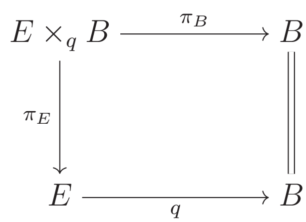
□

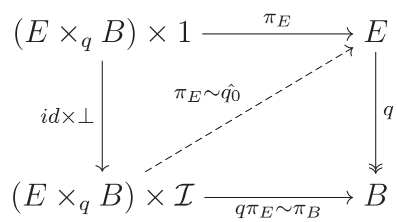
In fact, we can show that $\tilde{i}$ is an *exact inclusion* of path categories, in the sense that it preserves and reflects all the path category structure.

Theorem 6.18. *The functor $\tilde{i} : \mathsf{Eql} \to \mathbb{RT}(\mathbb{P})$ preserves and reflects the homotopy relation, fibrations, terminal object, and pullbacks along fibrations.*

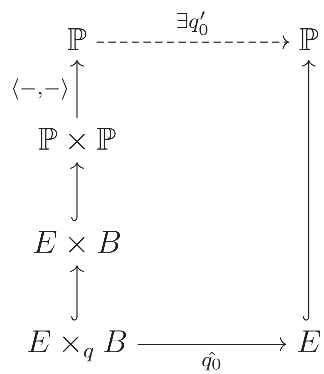
Proof. Let $f, g : X \to Y$ be two parallel maps in Eql. If $\tilde{i}(f), \tilde{i}(g)$ are $\mathbb{P}$-homotopic in $\mathbb{RT}(\mathbb{P})$, then for each $x \in X$, the set $Y(fx, gx)$ is non-empty and therefore $fx \sim gx$, so $f \sim g$. On the other hand, Suppose $f \sim g$. Then for each $x \in X$, we have $f_0 \xi_x = e_{B_Y}(fx) \in Y(fx, gx)$, and so f_0 codes a $\mathbb{P}$-homotopy. We can also take g_0 here and it doesn't matter. This shows $\tilde{i}$ preserves and reflects homotopy.

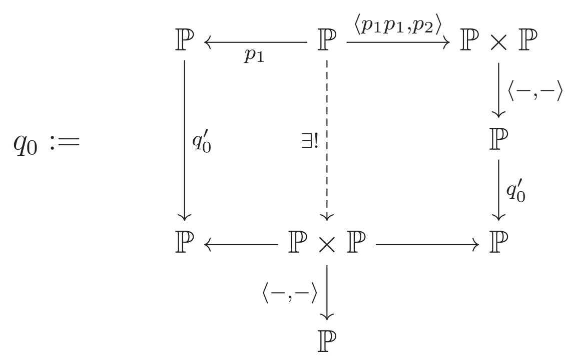
Let $q : E \to B$ in Eql. Suppose q is a $h_{\mathcal{I}}$-fibration. Then we need to show q has witnesses $q_0, q_1 : \mathbb{P} \to \mathbb{P}$ for its $\mathbb{P}$-fibration lifting property. We can construct q_0 by taking the following lefthand $\mathcal{I}$-homotopy pullback, i.e. the standard pullback in Equ, and then applying the $h_{\mathcal{I}}$-lifting property to it as on the righthand side:





inducing the transport map $\hat{q_0} : E \times_q B \to E$. To turn this into a map $\mathbb{P} \to \mathbb{P}$, we apply the extension theorem to particularly chosen embeddings as on the left hand side, so that q_0 may be defined as the righthand composite.





Here, $\langle\langle -, - \rangle\rangle$ is the pair-encoding homeomorphism, and $E \times B \hookrightarrow \mathbb{P} \times \mathbb{P}$ is the product of the embeddings for E and B that was chosen in defining

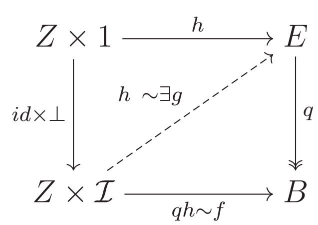
$\tilde{i}(E)$ and $\tilde{i}(B)$ respectively. Now, as we have defined it, q_0 satisfies, for any $e \in E, b \in B, \pi = \beta_{b'} \in B(q(e), b)$,

$$\begin{aligned} q_0 \langle\langle\langle\langle \epsilon_e, \beta_b \rangle\rangle, \beta_{b'} \rangle\rangle &= \langle\langle q_0' \langle\langle \epsilon_e, \beta_b \rangle\rangle, q_0' \langle\langle \epsilon_e, \beta_{b'} \rangle\rangle\rangle\rangle \\ &= \langle\langle e_{B_E}(\hat{q}_0(e,b)), e_{B_E}(\hat{q}_0(e,b'))\rangle\rangle \end{aligned}$$

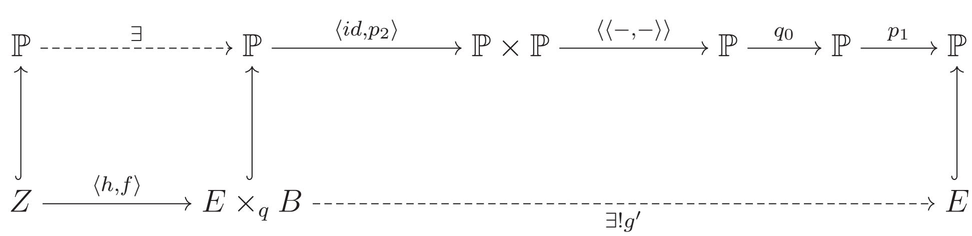
Taking $e' = \hat{q}_0(e,b)$ and $\rho = e_{B_E}(\hat{q}_0(e,b'))$, we have $q(e') = b$ and $q(\rho) = \pi$, thus showing that q_0 witnesses the lifting property of 1-cells for q.

Actually, q_0' also allows us to define q_1, since 1-cells of an equilogical space are just realizers of some element. We do not put the precise construction here, since it is just an exercise in composing pair encodings and decodings appropriately. This shows that q is also a $\mathbb{P}$-fibration in $\mathbb{RT}(\mathbb{P})$, so $\tilde{i}$ preserves fibrations.

On the other hand, suppose q is a $\mathbb{P}$-fibration, whose lifting properties are witnessed by q_0 and q_1. Now, we need to show q is a $h_{\mathcal{I}}$-fibration, i.e. that for any equilogical space Z in the following diagram, there is a map $g : Z \to E$ making commute

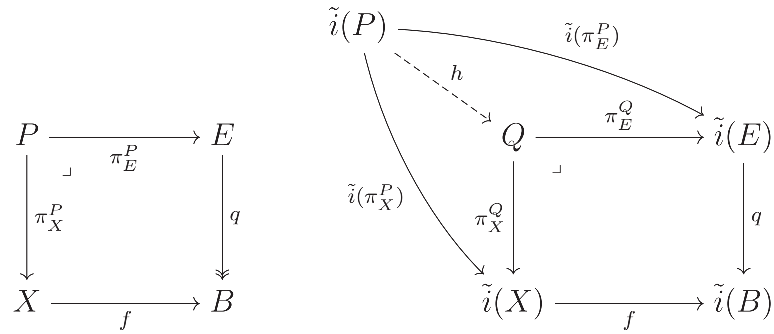


By the extension theorem, we can define the following composite map on top:

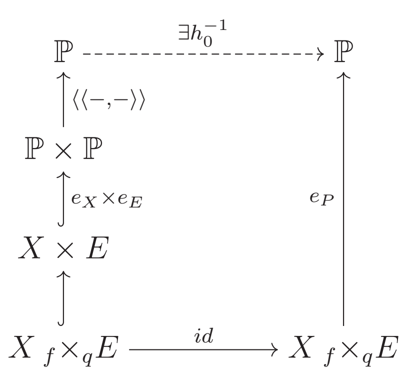


where the embedding of $E \times_q B$ is the special one we chose before. By the property of q_0 witnessing the lifting property, this composite on top restricts to the composite $g := g' \langle h, f \rangle$ below, and moreover we have that $\pi_E \sim g'$ and $qg' = \pi_B$. Hence, $h = \pi_E \langle h, f \rangle \sim g' \langle h, f \rangle = g$ and $qg = \pi_E \langle h, f \rangle = f$. This shows that fibrations are reflected by $\tilde{i}$.

Finally, we show that $\tilde{i}$ interacts well with pullbacks of fibrations. It is clear that $\tilde{i}$ reflects pullbacks since it is fully faithful. So it remains to show preservation of pullbacks. It suffices to show that the $\tilde{i}$-image of the standard pullback in Eql is isomorphic to the standard pullback in $\mathbb{RT}(\mathbb{P})$ of the $\tilde{i}$-image of the pullback diagram. So, suppose we have the standard pullbacks



with the induced map h. Notice that the underlying set/0-cells of P and Q are the same, so the map h is actually the identity on 0-cells. However, a witness h_0 for h must send the realizer $e_P(x, e)$ to $\langle\langle e_X(x), e_E(e)\rangle\rangle$ where e_P, e_X, e_E are the embeddings chosen for the definition of $\tilde{i}$. The same is true for the witness h_1 for 1-cells. Therefore, to show h is an isomorphism, it suffices to show that there is a witness h_0^{-1} for the inverse mapping. But this is essentially the identity map that only changes the choice of embedding, so we can obtain it by the extension theorem:



□

In light of this result, we will call $\mathbb{P}$-homotopies as $\mathcal{I}$-homotopy as well, just to reduce the number of homotopy relations we have to keep in mind.

Moreover, notice that the ability of $h_{\mathcal{I}}$-fibrations to lift whole homotopies at once, rather than just individual paths, corresponds to the computational uniformity of lifts for $\mathbb{P}$-fibrations in the previous proof. This suggests the fibrations of $\mathbb{RT}(\mathbb{P})$ can be thought of as a kind of Hurewicz fibration, and that removing the computational uniformity condition would correspond to *Serre fibrations*.

6.3 Homotopy Theory in Equ as an Amalgamation Problem

In Eql, since colimits are computed by actually gluing, there is no issue with defining the cut map for $\mathbf{I}$. The flip and contraction operations are also as usual. Moreover, since $\mathbf{I}$ is locally compact and Hausdorff, it is exponentiable in ωT_0 and hence exponentiable in Eql as well.

Proposition 6.19. *Let* $\mathbf{I}$ *be the standard unit interval with the discrete equivalence relation. It is an exponentiable interval object* $(\mathbf{I}, 0, 1, \tau, \kappa, \eta)$ *in* Eql*.*

Of course, $\mathbf{I}$ is not a strict interval object, and so for its associated path category structure we do not immediately know if $\langle s, t\rangle$ is an $h_{\mathbf{I}}$-fibration. However, the proof idea is essentially the same as for a strict interval, we just have to define the homotopy manually to avoid strictness issues.

Proposition 6.20. *For every equilogical space* X*, there is a factorization*

$$\begin{array}{ccccc} & & X^{\mathbf{I}} & & \\ & \overset{r}{\nearrow} & & \overset{\langle s,t\rangle}{\searrow} & \\ X & & \xrightarrow{\langle id, id\rangle} & & X \times X \end{array}$$

where r *is an* $\mathbf{I}$*-homotopy equivalence and* $\langle s, t\rangle$ *is an* $h_{\mathbf{I}}$*-fibration.*

Proof. As in the general case, $rt \simeq id_{X^{\mathbf{I}}}$ is witnessed by η.

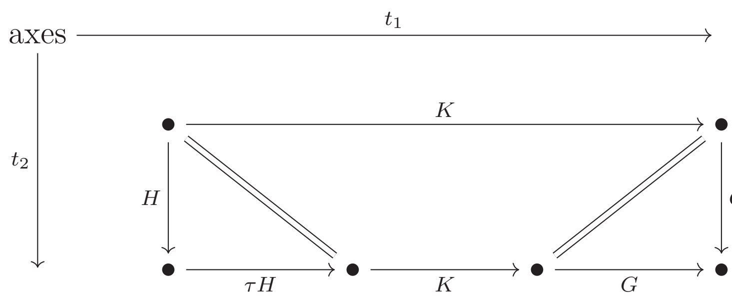
Next, to see that $\langle s, t\rangle$ is a h_I-fibration, consider the following lifting square:

$$\begin{array}{ccc} Z & \xrightarrow{K} & X^I \\ {\scriptstyle Z\times\bot}\downarrow & & \downarrow{\scriptstyle \langle s,t\rangle} \\ Z \times I & \xrightarrow[\langle H,G\rangle]{} & X \times X \end{array}$$

We define the filling homotopy $L : Z \times I \times I \to X$ as:

$$L(z, t_1, t_2) := \begin{cases} H(z, t_2 - 3t_1) & t_1 \leq \frac{1}{3}t_2 \\ K(z, \frac{1}{1-\frac{2}{3}t_2}(t_1 - \frac{1}{3}t_2)) & \frac{1}{3}t_2 \leq t_1 \leq 1 - \frac{1}{3}t_2 \\ G(z, 3t_1 + t_2 - 3) & 1 - \frac{1}{3}t_2 \leq t_1 \end{cases}$$

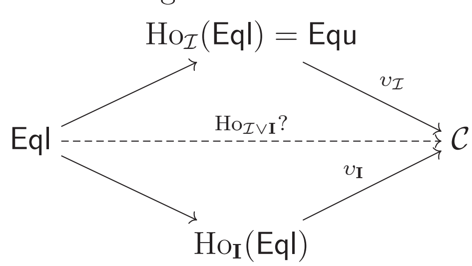
Graphically, L looks like:



Note that L is equivariant since each of H, K, G are. □

Theorem 6.21. *There is a path category structure on* Eql *whose fibrations are* $h_{\mathbf{I}}$*-fibrations and equivalences are* $\mathbf{I}$*-homotopy equivalences.*

We have now constructed two "homotopy theories" on Eql. But the existence of these two homotopy theories allow us to frame the search for a homotopy theory in Equ as an *amalgamation problem* of two homotopy theories. That is, we want a diagram that looks like the one below, where $\mathcal{C}$ is the homotopy category of some third notion of homotopy on Eql, in such a way that it factors through functors $\upsilon_{\mathcal{I}} : \mathsf{Equ} \to \mathcal{C}$ and $\upsilon_{\mathbf{I}} : \mathrm{Ho}_{\mathbf{I}}(\mathsf{Eql}) \to \mathcal{C}$. Ideally, these two functors would also be induced by some homotopy theories on their respective domain categories.



6.4 Length-global Homotopies

What should $\mathcal{C}$ be? If we take it as a pushout of the diagram in the previous section, then the objects of $\mathcal{C}$ are still equilogical spaces, but the maps are equivalence class of equivariant maps, up to alternating sequences of $\mathcal{I}$- and $\mathbf{I}$-homotopies. In other words, the notion of homotopy that should induce $\mathcal{C}$ is generated by interval objects $(\mathcal{I} \vee \mathbf{I})^n$. This is the length-global notion of homotopy. However, as we show in this section, neither a model structure nor a path category structure can be defined where the (weak) equivalences are length-global homotopy equivalences.

Non-existence of Model Structure on Equ

First, we show that there is no model structure on Equ for which the weak equivalences are length-global homotopy equivalences. The proof was originally[3] given by Tom goodwillie and Tim Campion for simplicial sets, where the interval object Δ^1 similarly induces a non-transitive homotopy relation. So, let $\simeq^*$ denote the transitive closure of the homotopy relation $\simeq_{\mathbf{I}}$ in Equ, which is precisely the length-global homotopy: $f \simeq^* g$ iff there is a sequence $(H_1 \dots H_n)$ of $\mathbf{I}$-homotopies, which are composable up to equivalence (i.e. $\mathcal{I}$-homotopies), s.t. $sH_1 \sim f$ and $tH_n \sim g$.

We can interpret this result as saying that there can be no homotopy localization functor $v_{\mathcal{I}}$, at least if we require the functor to be induced by a model structure on Equ.

Lemma 6.22. *In any model category on* Equ *for which the weak equivalences are $\simeq^*$-homotopy equivalences, the map $\emptyset \to 1$ is a cofibration.*

Proof. Factor $\emptyset \to 1$ as $\emptyset \rightarrowtail X \overset{\sim}{\twoheadrightarrow} 1$. Then $X \overset{\sim}{\twoheadrightarrow} 1$ has a homotopy inverse, so X is non-empty with a map $1 \to X$. Then we have a retract diagram

$$\begin{array}{ccccc} \emptyset & = & \emptyset & = & \emptyset \\ \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & X & \longrightarrow & 1 \end{array}$$

and so $\emptyset \to 1$ is also a cofibration. □

Theorem 6.23. *There is no model structure on* Equ *for which the weak equivalences are $\simeq^*$-homotopy equivalences.*

[3]https://mathoverflow.net/questions/462162/str%C3%B8m-model-structures-on-the-category-of-simplicial-sets

Proof. Suppose we have such a model structure. Consider the endpoint inclusion $t : 1 \to \mathbf{I}$, and factor it by $1 \overset{x_0}{\rightarrowtail} X \overset{\sim}{\twoheadrightarrow} \mathbf{I}$. Now, $p : X \overset{\sim}{\twoheadrightarrow} \mathbf{I}$ must be surjective, for otherwise, $\emptyset \to 1$ is an acyclic fibration by pulling back p over some point $x : 1 \to \mathbf{I}$ not in the image of p. Then, by the previous lemma we would be able to lift

$$\begin{array}{ccc} \emptyset & = & \emptyset \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow\sim \\ 1 & = & 1 \end{array}$$

which is a contradiction.

Now, since p is surjective into a discrete equilogical space $\mathbf{I}$ with more than two points, there must be another point $x_1 : 1 \to X$ which is not equivariant to x_0. Now, note that by our definition of weak equivalences, I is contractible, so also X is contractible. But from this, the following argument creates a contradiction:

1. First, note that x_0 is acyclic since X is contractible. That is, $X \to 1$ has a homotopy inverse $x : 1 \overset{\sim}{\to} X$ with $X \to 1 \overset{x}{\to} X \simeq^* id_X$. Then also x_0 has homotopy inverse $X \to 1$:

$$X \to 1 \overset{x_0}{\rightarrowtail} X \simeq^* X \to 1 \overset{x_0}{\rightarrowtail} X \to 1 \overset{x}{\to} X = X \to 1 \overset{x}{\to} X \simeq^* id_X$$

2. Viewing (X, x_0, x_1) as an interval object, construct the transfinite composition

$$1 \overset{x_0^1}{\to} X \overset{x_0^2}{\to} X^2 = X \vee X \overset{x_0^3}{\to} X^3 = (X \vee X) \vee X \ldots \to X^\infty$$

where each x_0^i is an acyclic cofibration obtained by pushing out x_0:

$$\begin{array}{ccc} 1 & \longrightarrow & X^{i-1} \\ {\scriptstyle x_0 \sim}\downarrow & & \downarrow{\scriptstyle \sim x_0^i} \\ X & \longrightarrow & X^i \end{array}$$

Then the transfinite composite $1 \to X^\infty$ is also an acyclic cofibration, since acyclic cofibrations are closed under transfinite composition[4].

[4] https://ncatlab.org/nlab/show/injective+or+projective+morphism#ClosurePropertiesOfInjectiveAndProjectiveMorphisms

This constructs a contraction of the infinite join X^∞ to the x_0 point in the first copy of X within X^∞.

3. By definition of homotopy, this means there is a finite length n such that for every point x in X^∞ there is an $\mathbf{I}$-path of length n between x and x_1. However, recall that the pushout in Equ only adds new equivalences without actually gluing the underlying topological spaces. Hence, it takes at least $n+1$ $\mathbf{I}$-paths for a point in the $n+2$th copy of X to reach the x_0 point in the first copy of X. This is a contradiction.

□

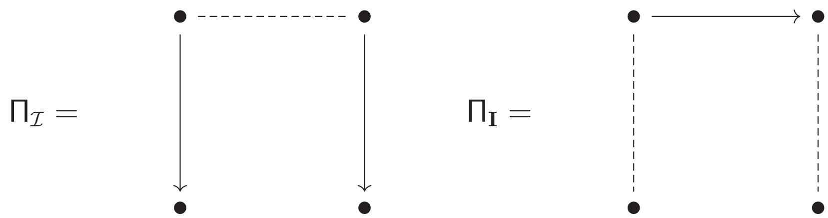
Essentially, the proof uses the $(\mathsf{Cof}, \mathsf{W} \cap \mathsf{Fib})$-factorization to construct a space X which contracts into an acyclic cofibrant point. Then infinitary closure properties of acyclic cofibrations allow us to chain together infinitely many copies of X, while keeping it contractible. This collides against the inherent finiteness in our definition of homotopy. However, in the proof we made essential use of the fact that colimits in Equ are defined by adding equivalence relations. This means that the proof would not work in Eql. In the following subsection, we look at Eql.

Amalgamating Path Categories in a subcategory of Eql

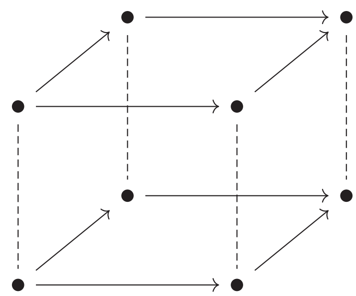
In this subsection, we observe that there is no path category structure on Eql for which the equivalences are length-global homotopy equivalences, at least if we require certain reasonable conditions. Let us denote $\mathbb{I} = \mathcal{I} \vee \mathbf{I}$ for ease of reference.

The problem is in assigning a path object PY. For a length n $\mathbb{I}$-homotopy between $f, g : X \to Y$, its corresponding path object should be $Y^{\mathbb{I}^n}$. Hence, each $Y^{\mathbb{I}^n}$ should be a path object in our path category. However, in [BM18, Corollary 2.10] it is shown in an arbitrary path category that if there is a homotopy between f, g with respect to one path object PY, then there is a homotopy f, g with respect to *any* path object $P'Y$. In our context, this implies that if such a path category exists, then whenever f, g are length n homotopic, then they are already length 1 homotopic. This is clearly not true in general.

Regardless, we can try to restrict to a subcategory in which this statement is true. We would like $X^{\mathbb{I}}$ to be a path object, which would in particular require $\langle s, t \rangle$ to be a fibration. At the very least a fibration in our desired amalgamating path category should be both an $h_{\mathbf{I}}$- and $h_{\mathcal{I}}$-fibration. However, $\langle s, t\rangle : X^{\mathbb{I}} \to X \times X$ being both a $h_{\mathbf{I}}$- and $h_{\mathcal{I}}$-fibration essentially requires X to be able to fill in *horns* of the form



where the solid line is a $\mathbf{I}$-homotopy and the dashed line is a $\mathcal{I}$-homotopy. Clearly, not all equilogical spaces can do this so restrict to those spaces that can by adding these lifting conditions into the definition of a fibration, and restricting to the fibrant objects. But then, to show $\langle s, t\rangle$ satisfies the new definition of fibration, i.e. that it can lift $\sqcap_{\mathbf{I}}$, amounts to lifting a three-dimensional horn $\sqcap^2_{\mathbf{I}}$, i.e. the following cube with the bottom face removed:



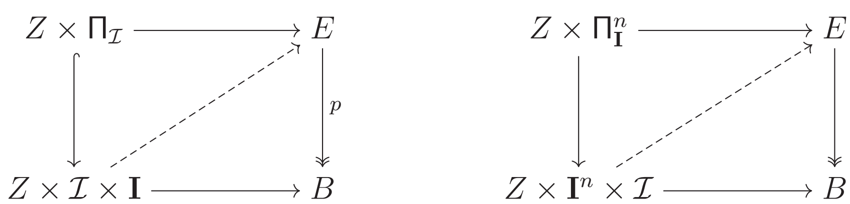
But then if we add this into the fibration condition, then $\langle s, t\rangle$ being a fibration amounts to lifting the corresponding four-dimensional horn $\sqcap^3_{\mathbf{I}}$, etc. So, we are lead to adding in some kind of *Kan cubical conditions* to our fibrations, ensuring that the two notions of homotopy interact well. One might wonder why we don't have to add corresponding higher dimesional $\sqcap^n_{\mathcal{I}}$. This is because $\mathcal{I}$ is so degenerate that the higher dimensional horn filling conditions are implied by the 2-dimensional condition, as we will see.

Definition 6.24. The *mixed* $\mathbf{I}$-*horn* shape of dimension $n \geq 1$ is the following colimit computed in Equ:

$$\begin{array}{ccc} \partial\mathbf{I}^n & \longrightarrow & \mathbf{I}^n \\ \| & \mathrm{Ho}_{\mathcal{I}} & \downarrow \\ \partial\mathbf{I}^n & \longrightarrow & \sqcap^n_{\mathbf{I}} \end{array}$$

where $\partial\mathbf{I}^n$ is the boundary shape of $\mathbf{I}^n$. The mixed $\mathcal{I}$-horn shape is $\sqcap_{\mathcal{I}}$ as portrayed above.

A map $p : E \to B$ is a *mixed fibration* if it is both a $h_{\mathcal{I}}$ and $h_{\mathbf{I}}$ fibration, and additionally, for any equilogical space Z in any commutative square of the forms below, there is a lift.



Let Eql_f denote the full subcategory of Eql containing the mixed-fibrant equilogical spaces. ♢

Remark 6.25. Notice that $\mathbb{I}$ is not itself mixed fibrant, but $\mathbf{I}$ and $\mathcal{I}$ are, since there are no non-trivial mixed horns in $\mathbf{I}$ or $\mathcal{I}$. This is not so strange: in simplicial sets, the interval object Δ^1 is also not a Kan complex. However, this does mean we cannot generate the path category from the interval object since it doesn't actually exist in Eql_f. ◎

In the following proposition we observe that a mixed-fibrant space is essentially a "boring" equilogical space whose equivalence relation has to be the relation that induces the connected components $\pi_0(|X|)$ of the underlying space. This rules out a lot of equilogical spaces, which quantifies the price we pay for the path category. In particular, neither $\mathbf{I}$ nor $\mathcal{I}$ nor $\mathbb{I}$ are themselves mixed-fibrant.

Proposition 6.26. *If an equilogical space X is mixed-fibrant, then for any equivariant maps $f, g : Y \to X$ from any equilogical space Y,*

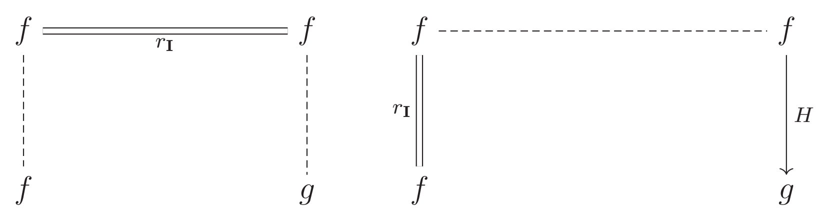
$$f \simeq_{\mathcal{I}} g \iff f \simeq_{\mathbf{I}} g.$$

In particular, for any $x, x' \in |X|$,

$$x \sim_X x' \iff \exists p : \mathbf{I} \to X. p(0) = x \text{ and } p(1) = x'.$$

Proof. Suppose X is mixed-fibrant. Then with the assumed relation given above in the following diagram, we can fill the horns below to get the other relation.

$f \sim g$ $\qquad\qquad$ $\exists H : f \simeq_{\mathbf{I}} g$



□

This means we do not even have to take the transitive, symmetric closure of $\mathbb{I}$-homotopy anymore, since it is already an equivalence relation.

Corollary 6.27. *$\mathbb{I}$-homotopy is a congruence relation in Eql_f.*

Proof. A $\mathbb{I}$-homotopy is just a $\mathbf{I}$-homotopy followed by a $\mathcal{I}$-homotopy. By the previous correspondence and transitivity of $\mathbf{I}$-homotopy, the $\mathbb{I}$-homotopy relation on mixed-fibrant spaces is the same as the $\mathbf{I}$-homotopy relation and the $\mathcal{I}$-homotopy relation, which are both congruences. □

Corollary 6.28. *Any $\mathbb{I}$-acyclic mixed-fibration of mixed-fibrant objects $p : E \to B$ has a section that is also a $\mathbb{I}$-homotopy inverse.*

Proof. A $\mathbb{I}$-homotopy equivalence of mixed-fibrant objects is a $\mathbf{I}$-homotopy equivalence. Therefore, p is an acyclic fibration in the $\mathbf{I}$-induced path category structure on $\mathbf{I}$, and so has a section that is also a $\mathbf{I}$-homotopy inverse. But then this section is also a $\mathbb{I}$-homotopy inverse. □

Now, the previous two corollaries give us axiom (PC3) for equivalences and axiom (PC4) for acyclic fibrations. We also have axiom (PC2) for fibrations since the fibrations are just defined by lifting properties so they must be closed under pullbacks. This leaves axiom (PC1). As we will see, for a mixed-fibrant space X, both $X^{\mathcal{I}}$ and $X^{\mathbf{I}}$ are path objects.

Proposition 6.29. *For any mixed-fibrant equilogical space X and equilogical space Y, if X^Y exists then it is mixed-fibrant. If Y is also mixed-fibrant then $X \times Y$ is mixed-fibrant.*

Proof. The first statement is clear by using the adjunction to put Y on the left side of the lifting diagrams for the mixed-fibrant conditions. The second statement also follows by lifting for X and Y separately and using the universal property of the product. □

Proposition 6.30. *For any mixed-fibrant equilogical space X, we have path objects*

$$\begin{array}{ccc} & X^{\mathcal{I}} & \\ {\scriptstyle\sim}\nearrow & & \searrow\!\!\!\!\to \\ X & \longrightarrow & X \times X \end{array} \qquad \begin{array}{ccc} & X^{\mathbf{I}} & \\ {\scriptstyle\sim}\nearrow & & \searrow\!\!\!\!\to \\ X & \longrightarrow & X \times X \end{array}$$

Proof. Every $\mathcal{I}$- and $\mathbf{I}$-homotopy equivalence is a $\mathbb{I}$-homotopy equivalence, so it is clear that the left factors of both path objects are $\mathbb{I}$-homotopy equivalences.

Now, to prove that $\langle s_{\mathbf{I}}, t_{\mathbf{I}} \rangle$ is a mixed fibration. We have already proven it is a $h_{\mathbf{I}}$-fibration, and a $h_{\mathcal{I}}$ lifting diagram for $\langle s, t \rangle$ corresponds to a $\sqcap^1_{\mathbf{I}}$-horn lifting property for X:

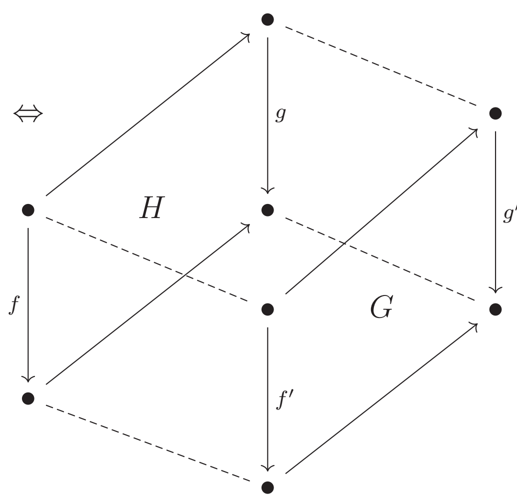
$$\begin{array}{ccc} Z & \xrightarrow{H} & X^{\mathbf{I}} \\ {\scriptstyle Z\times\bot}\downarrow & & \downarrow{\scriptstyle\langle s,t\rangle} \\ Z \times \mathcal{I} & \xrightarrow[\langle f \simeq_{\mathcal{I}} f', g \simeq_{\mathcal{I}} g' \rangle]{} & X \times X \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} f & \xrightarrow{H} & g \\ \vdots & & \vdots \\ f' & & g' \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ccc} Z \times \sqcap^1_{\mathbf{I}} & \xrightarrow{f' \sim f H g \sim g'} & X \\ \big\downarrow & & \big\downarrow \\ Z \times \mathbf{I} \times \mathcal{I} & \longrightarrow & 1 \end{array}$$

so by the mixed-fibrancy of X we can fill such a lifting diagram. In fact, we can see that the $h_{\mathcal{I}}$ lifting property corresponds to a $\sqcap^0_{\mathbf{I}}$-horn lifting. This proof then generalizes: any $\sqcap^n_{\mathbf{I}}$-horn for $\langle s, t \rangle$ is a $\sqcap^{n+1}_{\mathbf{I}}$-horn for X, which we can fill. Finally, a $\sqcap_{\mathcal{I}}$-horn for $\langle s, t\rangle$ corresponds to the following shape:

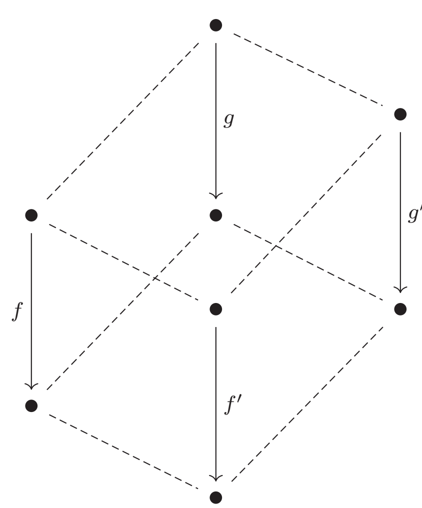
$$\begin{array}{ccc} Z \times \sqcap_{\mathcal{I}} & \xrightarrow{[H,G]} & X^{\mathbf{I}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \langle s,t\rangle} \\ Z \times \mathcal{I} \times \mathbf{I} & \xrightarrow[\langle f\sim f', g\sim g'\rangle]{} & X \times X \end{array} \qquad \Leftrightarrow$$



In order to fill this diagram, we need to show that $H \sim G$. We can do this by observing that for each $t \in \mathbf{I}$,the z-axis slice $[H(t, -), G(t, -)]$ of the above diagram forms a $\sqcap_{\mathcal{I}}$-horn in X, whose filler shows that $H(t, -) \sim G(t, -)$.

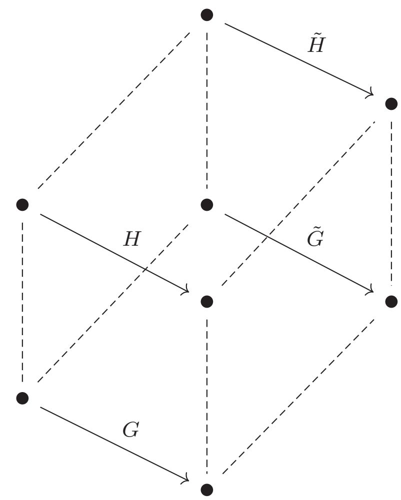
Finally, let us prove that $\langle s_{\mathcal{I}}, t_{\mathcal{I}}\rangle$ is a mixed fibration. It is already a $h_{\mathcal{I}}$-fibration, and as before the $h_{\mathbf{I}}$-fibration condition for $\langle s, t\rangle$ corresponds to a $\sqcap_{\mathcal{I}}$-horn on X which we can fill. We observe that a $\sqcap_{\mathcal{I}}$-horn for $\langle s, t\rangle$ already contains the necessary data and properties of its filler, so this horn is trivially fillable:

$$\begin{array}{ccc} Z \times \sqcap_{\mathcal{I}} & \xrightarrow{[f\sim g],[f'\sim g']} & X^{\mathcal{I}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \langle s,t\rangle} \\ Z \times \mathcal{I} \times \mathbf{I} & \xrightarrow[\langle f\sim f', g\sim g'\rangle]{} & X \times X \end{array} \qquad \Leftrightarrow$$



For a $\sqcap^n_{\mathbf{I}}$-horn in $\langle s,t\rangle$, we have the following diagram (noting that the righthand diagram portrays I^n as a single dimension):

$$\begin{array}{ccc} Z \times \sqcap^n_{\mathbf{I}} & \xrightarrow{(sG\sim H\sim tG)\sim(s\tilde{G}\sim\tilde{H}\sim t\tilde{G})} & X^{\mathcal{I}} \\ \downarrow & & \downarrow{\scriptstyle \langle s,t\rangle} \\ Z \times \mathbf{I}^n \times \mathcal{I} & \xrightarrow[\langle [H,G],[\tilde{H},\tilde{G}]\rangle]{} & X \times X \end{array} \quad \Leftrightarrow$$



and to fill this it suffices to show $G \sim \tilde{G}$. However, we can do this pointwise, i.e. for each $t \in \mathbf{I}^{n-1}$, we have the $\sqcap_{\mathcal{I}}$-horn $[G(t,-),\tilde{G}(t,-)]$ in X. Filling this shows $G(t,x) \sim \tilde{G}(t,x)$ for all $x \in \mathbf{I}$, which suffices to show $G \sim \tilde{G}$. $\square$

Theorem 6.31. *There is a path category structure on* Eql_f *whose equivalences are* $\mathbb{I}$*-homotopy equivalences and the fibrations are mixed fibrations.*

Indeed, we have developed a path category structure, but it required us to restrict to a vanishingly small subcategory of boring equilogical spaces. We end this investigation into length-global homotopies by remarking that there is yet another notion of homotopy theory called Λ-fibration categories [Min00; Min02] in which the definition explicitly accomodates the use of a Λ-indexed family of path objects where Λ is "almost" a filtered category. We have verified that the $Y^{\mathbb{I}^n}$ may be arranged into such a family where the objects of Λ are lengths n. However, we leave the investigation of putting a Λ-fibration structure for equilogical spaces as future work, since there are multiple, rather complicated notions of homotopy equivalence at play in this framework.

6.5 Length-local Homotopies

In the previous section, we explored the length-global notion of homotopy, which does not seem to behave well at all when we try to fit it into a path category structure or a model structure. In this section, we will briefly shift focus to the length-local notion, and outline a possible strategy for obtaining a length-local path category structure for Equ.

Based on the cocylindrical view of homotopies, a length-local homotopy can be defined as a map $H : X \to PY$ for some cocylinder PY consisting of all the $\mathbb{I}^n$-paths in Y. Then $H(x)$ and $H(x')$ can have different lengths for $x \neq x'$. So unlike the length-global case, it should, at first glance, be possible to have a single such path object, and thus avoiding the troubles of the previous section.

However, the problem with this definition in terms of a path object is that it doesn't a priori give a notion of length-local homotopy, which would yield a universal property that we can use to check whether our construction of PY is "correct". In the absence of such a universal property, we follow axioms provided in [BG12] for the notion of a Moore structure, fully stated in Appendix B. A Moore structure is, briefly, a functorial and pullback-preserving cocylinder M with a path contraction map $M \to MM$ and a length map $MY \to M1$, where we view $M1$ as the "object of path lengths". If $M1 \cong 1$, i.e. M preserves finite limits, then we call such a Moore structure a *nice* Moore structure, following [Doc14]. The axioms for (nice) Moore structures are also quite well motivated in [Doc14].

In Appendix B, we additionally document some difficulties in directly constructing a Moore structure in Equ, with the *assumption* that the underlying space of MX should be some space of *continuous* maps $\mathbb{I}^n \to X$. While the difficulty boils down to a seemingly technical conflict between the continuity of the path contraction operation and pullback preservation, in fact this appears to be a *false* assumption. This assumption stems from the view of equilogical spaces as spaces first, and computational representations second, which follows from the motivation given in the introduction. We further discuss the issues with this view in the following discussion chapter.

The construction is clarified once we consider equilogical spaces as computational representation, i.e. consider them in the wider context of the $\mathsf{RT}(\mathbb{P})$ and $\mathbb{RT}(\mathbb{P})$. Following the same lines of thought as in [Van15] and [JO20], but using $\mathbb{RT}(\mathbb{P})$ to clarify some ideas, we outline a proof strategy for obtaining a Moore structure in Equ.

In [Van15], a Moore structure for a length-local notion of homotopy is given in the effective topos, where $\mathbf{I}$ is replaced by the object with two elements sharing a common realizer. The construction of the Moore structure is done by stitching together the exponential objects $Y^{(\mathcal{I} \vee \mathbf{I})^n}$.

Therefore, we wish to understand this exponential. This can be done by instead understanding homotopy exponentials in $\mathbb{RT}(\mathbb{P})$, which correspond to exponentials in $\mathsf{RT}(\mathbb{P})$.

Definition 6.32 ([Ber20, Definition 2.16])**.** Let $\mathcal{C}$ be a path category. Then for any two objects X and Y, we will call an object X^Y a homotopy exponential of X and Y if there is a map $\mathrm{ev} : X^Y \times Y \to X$ such that for any map $h : A \times Y \to X$ there is a unique-up-to-homotopy map $H : A \to X^Y$ such that $\mathrm{ev}(H \times id_Y) \simeq h$. ♢

Remark 6.33. It is clear that under the homotopy functor $P : \mathbb{RT}(\mathbb{P}) \to \mathsf{RT}(\mathbb{P})$ the homotopy exponential yields the exponential object, provided the product used in the above definition is a homotopy product. However, since every object in a path category is fibrant, in fact the usual product is a homotopy product, since pullbacks of fibrations are homotopy pullbacks. ⊚

Proposition 6.34 ([Ber20, Proposition 4.1])**.** *Let A, B be objects in $\mathbb{RT}(\mathbb{P})$. The homotopy exponential A^B in $\mathbb{RT}(\mathbb{P})$ can be constructed with the following details.*

- *The* 0*-cells are triples (f, f_0, f_1) where f_0 and f_1 witnesses $f : B \to A$ as a morphism in $\mathbb{RT}(\mathbb{P})$, and the realizer is $\phi_{(f,f_0,f_1)} = \langle\langle f_0, f_1\rangle\rangle$.*
- *The* 1*-cells $A^B(f, g)$ is the set of coded homotopies between f and g.*

We will define the Moore structure by putting elements of $A^{\mathbb{I}^n}$ together in one object, and 1-cells between paths of length n and m are coded homotopies up to *subdivision* of intervals.

Definition 6.35. Let us denote points of $|\mathbb{I}^n|$ by $kI + t$ for the element t in the $(k + 1)$-th component of $\mathbb{I}^n$, and 0 as the first element. For example, $|\mathbb{I}^2|$ is denoted

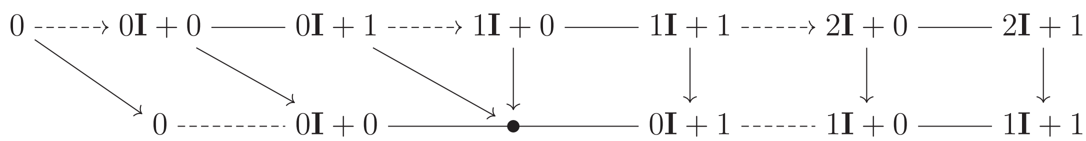
$$0 \text{ - - - - - - } 0\mathbf{I} + 0 \text{ ——— } 0\mathbf{I} + 1 \text{ - - - - - - } 1\mathbf{I} + 0 \text{ ——— } 1\mathbf{I} + 1$$

Now, there is a natural lexicographic ordering on these elements. A *subdivision map* then is an endpoint- and order-preserving map $\mathbb{I}^m \to \mathbb{I}^n$ in Eql[5]. ♢

Remark 6.36. Concretely, every subdivision map can be described by picking $m - n$ points on $\mathbb{I}^n$, and then defining the $2m + 1$-element list consisting of the endpoints of each component, along with two copies of each chosen

[5]which we can of course consider as a map in $\mathbb{RT}(\mathbb{P})$ by the fully faithful embedding $\tilde{i}$

point, in the natural ordering. Then this identifies each of the $2m+1$ endpoints-of-components in $\mathbb{I}^m$ with elements of the previous list, which in turn identifies where to map the 0 and each component of $\mathbb{I}^m$ to. Then, picking an order- and endpoint- preserving continuous map $\mathbf{I} \to \mathbf{I}$ for each component of $\mathbb{I}^m$ precisely defines the subdivision map, for example, for $m = 3$ and $n = 2$, we can pick the bullet point $\bullet$, which determines where each component of $\mathbb{I}^3$ lands. Then it suffices to specify the behavior of the map on each component separately.



(Here, we portray the components with uneven lengths to save space.) ⊚

Definition 6.37. Let $A \in \mathbb{RT}(\mathbb{P})$. The Moore structure $M'A$ is defined by:

- The 0-cells are tuples (n, f, f_0, f_1) where f_0, f_1 witness $f : \mathbb{I}^n \to A$ and $\mu_{n,f,f_0,f_1} = \langle\langle f_0, f_1\rangle\rangle$.
- The 1-cells $M'A((n, f), (m, g))$ are coded homotopies p between f and $g\sigma$ or $f\sigma$ and g, for some subdivision map $\sigma : \mathbb{I}^n \to \mathbb{I}^m$ or $\sigma : \mathbb{I}^m \to \mathbb{I}^n$ respectively.

◊

Notice that this violates the aforementioned assumption. The 0-cells are not just continuous maps f, but they include witnesses f_0, f_1 of continuity of the map. At this point, to define the Moore structure in $\mathsf{RT}(\mathbb{P})$, we will need to extend the construction of essential surjectivity in the proof of Proposition 6.15 into a functor $Q : \mathsf{RT}(\mathbb{P}) \to \mathbb{RT}(\mathbb{P})$, so that the Moore structure can be defined by $M := PM'Q$. Then, we note that the inclusion functor $i : \mathsf{Equ} \to \mathsf{RT}(\mathbb{P})$ has a left adjoint, so is limit preserving. From this point, we need to

1. Construct all the required structure on M, which conceivably can be done on M' and then induced to M.
2. Show that $Mi(X)$ is isomorphic to an equilogical space in $\mathsf{RT}(\mathbb{P})$, for any equilogical space X. Alternatively, show that $M'\tilde{i}(X)$ is $\mathcal{I}$-homotopic to an equilogical space.

3. Show that M preserves pullbacks in $\mathsf{RT}(\mathbb{P})$. This may be doable on M' in $\mathbb{RT}(\mathbb{P})$ by considering $\mathcal{I}$-homotopy pullbacks. One complication is that homotopy pullbacks do not generally correspond to pullbacks in the homotopy category, and the latter is the second notion we are really after.

Item 1 requires a lot of busy work, while item 3 requires further understanding of the homotopical relationship between $\mathbb{RT}(\mathbb{P})$ and $\mathsf{RT}(\mathbb{P})$, which we consider to be beyond the scope of this thesis. However, we can quickly address item 2. Let us first observe characteristics of an object $\tilde{i}(X)$.

Definition 6.38 ([Ber20, Definition 5.1])**.** An object (A, α) in $\mathbb{RT}(\mathbb{P})$ is *standard discrete* if $\alpha_a = \alpha_{a'}$ implies $a = a'$ for $a, a' \in A$, i.e. α is injective. $\diamondsuit$

Clearly, any object $\tilde{i}(X)$ is standard discrete, since its realizer map is the given embedding $e_B : X \hookrightarrow \mathbb{P}$. We also see that $M'A$ is always standard discrete, since the realizers f_0, f_1 determine the map f. However, $\tilde{i}(X)$ has the additional property that its 1-cells correspond to realizers of 0-cells. We can express this by constructing an adjoint to $\tilde{i}$.

Definition 6.39. Given a standard discrete object (A, α), we define its *equification* $E(A)$ as the equilogical space $(A, \sim_A)$ where $a \sim_A a'$ iff $A(a, a') \neq \emptyset$. The topology on A is induced by the injection $\alpha : A \hookrightarrow \mathbb{P}$. $\diamondsuit$

Proposition 6.40. *Let X be an equilogical space. Then $M'\tilde{i}X$ is $\mathcal{I}$-homotopy equivalent to $\tilde{i}E(M'\tilde{i}X)$.*

Proof. Let $A = M'\tilde{i}X$ and $B = \tilde{i}E(M'\tilde{i}X)$. Then we will define two maps $m : A \to B$ and $n : B \to A$, which are both the identity on 0-cells $|A|$. We note that the realizer maps α and β are different, since the latter was independently chosen when defining $\tilde{i}$. Nevertheless, by the extension theorem of $\mathbb{P}$ we have maps $m_0, n_0 : \mathbb{P} \to \mathbb{P}$ witnessing the change of embeddings.

For all $(n, f), (m, g) \in |A|$, the witness m_1 has to produce, given the tuple $\langle\langle f_0, f_1, g_0, g_1, p\rangle\rangle$, the realizers of some (k, h) such that $(n, f) \sim_B (k, h) \sim (m, g)$, but for this we can just take $h = f$, and take m_1 to be the projection to $\langle\langle f_0, f_1\rangle\rangle$.

On the other hand, n_1 has to produce, given the tuple $\langle\langle \beta_{(n,f)}, \beta_{(m,g)}, \beta_{(k,h)}\rangle\rangle$ for which $(n, f) \sim_B (k, h) \sim_B (m, g)$, some coded homotopy between f and $g\sigma$ (or $f\sigma$ and g). Given $(n, f) \sim_B (m, g)$, we know such a coded homotopy exists, but since we are looking for a coded homotopy between maps of equilogical spaces, we see that f_0 provides such a coded homotopy. So n_1 just has to project to $\beta_{(f,n)}$ and apply n_0 to obtain f_0.

Then it is clear that these two maps are $\mathcal{I}$-homotopy inverses, since they are the identity on 0-cells. That is, we need to exhibit coded homotopies

$$(f, n) \mapsto A((f, n), (f, n)) \text{ and } (f, n) \mapsto B((f, n), (f, n))$$

but this can clearly be given by the corresponding identity realizer $\mathbf{i}$ for A and B. □

This shows that M' of an equilogical space is itself an equilogical space up to $\mathcal{I}$-homotopy, and moreover we have provided the construction of this equilogical space. This provides us the seed for constructing the Moore structure on Equ.

Chapter 7

Discussion

Let us recap the developments in this thesis, and review how well they have achieved the desired outcome. Based on this, we also identify further work to be done.

Effective morphisms of Equilogical Spaces

We have introduced the QCB spaces as a notion of computationally representable space, and the equilogical spaces as computational representations of QCB spaces. This is formally presented as a quotienting functor $L : \mathsf{Equ} \to \mathsf{QCB}$. We further showed that every QCB space (armed with a choice of pseudobase enumeration) has a standard, well-behaved representation, and this is encapsulated as a fully faithful right adjoint $R : \mathsf{QCB} \to \mathsf{Equ}$.

In fact, R allows us to view QCB as a reflective subcategory of Equ, but I think this view is hampered by the need to choose a pseudobase enumeration. That is, the pseudobase enumeration simply encodes the data of the representation in a minimalist way, so what we are doing is really showing that the category of pairs (X, B) - where B is a pseudobase enumeration for the QCB space X - reflectively embeds into Equ. Regardless, there is still much merit to the existence of R, because it shows that every continuous map of QCB spaces is represented by some map between some representations. This also holds for representations over Kleene's second model, and in [BSS07] this is taken to mean that every continuous map of QCB spaces is "physically feasible".

Of course, for representation in equilogical spaces this is less justified because morphisms of equilogical spaces don't carry any *effective/recursive* computational content. But this begs the question: can we consider a wide subcategory of the equilogical spaces whose maps are effectively computable/feasible? By considering the framework of *relative realizability*, the answer is yes, but its not quite a subcategory.

In relative realizability, we consider the subset $\mathbb{P}_\#$ of $\mathbb{P}$ consisting of the recursively enumerable subsets of $\mathbb{N}$. Then the category of relative modest sets $\mathsf{Mod}(\mathbb{P}, \mathbb{P}_\#)$ has maps as functions realized by recursively enumerable subsets. For the corresponding category $\mathsf{Equ}_\#$ of relative equilogical space, the choice of basis enumeration has to be made explicitly part of the data of the object, because the morphisms are (equivalence classes of) continuous maps realized by recursively enumerable graphs [Bau00] that have to interact with the basis enumeration. It would be interesting to see whether every QCB map is actually representable in $\mathsf{Equ}_\#$, since this would establish the maps in $\mathsf{Equ}_\#$ as computational representations of maps between QCB spaces.

Homotopical Realizability

In section 6.5, we briefly noted that the desire to view equilogical spaces as spaces by sweeping away the choice of the basis enumeration lead us down the wrong path in constructing the length-local path object. Indeed, once we try to incorporate effectiveness conditions into the morphisms, this view falls apart. We therefore revise this intuition. An equilogical "space" is a set X equipped with a computational presentation of a topology on X. Morphisms in Equ are functions with a witness of continuity. Morphisms in $\mathsf{Equ}_\#$ are functions with a witness of continuity and effectiveness. We must view $\mathbb{P}$ as a model of "computation" where computable corresponds to continuity, not effectiveness.

Once we view equilogical spaces not as spaces but as computational representations from the standpoint of realizability (by embedding Equ in $\mathsf{RT}(\mathbb{P})$ and Eql in $\mathbb{RT}(\mathbb{P})$), we arrived at a promising construction for the length-local path object. However, we left the proof of correctness for this path object open, since it would require even further development of the homotopical analysis of realizability. I believe that further developing this analysis is the immediate future work for this thesis.

On a more foundational level, it seems that the identity of objects in $\mathsf{RT}(\mathbb{P})$ are rather elusive, i.e. it is quite unclear when two objects are isomorphic. With $\mathbb{RT}(\mathbb{P})$, this is clarified as being a $\mathcal{I}$-homotopy equivalence. Therefore, I believe that the homotopical viewpoint from $\mathbb{RT}(\mathbb{P})$ can complement the topos-theoretic viewpoint in $\mathsf{RT}(\mathbb{P})$, and it is imperative that the relationship between the two categories be fleshed out.

Miscellaneous Future Work

Beyond the considerations above, we also consider the following prospects.

Alternative Presentation of the Reals. In [Bau00], it is noted that from an internal logic perspective the reals $(\mathbb{R}, =)$ with the identity relation is very poorly behaved. Rather, one should consider the space of rapidly converging rational cauchy sequences, with two such sequences being equivalent if they identify the same real number. It would be interesting to explore the corresponding representation for the interval, and to see if this might have a better homotopy theory in Equ or Eql.

Simplicial characterization of homotopy theory in QCB. On the category of topological spaces, there is a *Quillen model structure* for which the weak equivalences are weak homotopy equivalences, i.e. maps that induce isomorphisms on homotopy groups [Qui67]. This model structure is equivalent to the classical model structure, and moreover is equivalent to the Quillen model structure on simplicial sets, which are combinatorial structures intended to represent the abstract structure of simplices in a space. This equivalence exposes the essentially combinatorial nature of homotopy theory on topological spaces, leading to a better understanding.

For QCB, we found it difficult to construct a Quillen model structure since the proof of the factorization axiom requires the use of uncountable colimits, which we do not have in QCB. However, we can still try to establish directly an equivalence between the classical model structure on QCB and some sub-model-category of simplicial sets. It is our conjecture that the model structure on *countable* simplicial sets is equivalent to the classical model structure on QCB. This would give us a combinatorial understanding of homotopy theory on QCB spaces. As a starting observation, every countable simplicial set can be geometrically realized as a QCB space, since such a realization would only involve countable colimits.

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Appendix A

A Characterization of the ωT_0 CW Complexes

In this chapter, we characterize the CW complexes in ωT_0, as a demonstration of how the lack of colimits in ωT_0 impacts our ability to carry out algebraic topology in the category. In particular, we characterize the CW complexes with a countable basis, since all CW complexes are T_0.

A.1 Additional Preliminaries

CW complexes are spaces obtained by gluing disks together.

Definition A.1. Let $n \geq -1$. The circle S^n is defined by $S^n = \{ x \in \mathbb{R}^{n+1} \mid ||x|| = 1 \}$. The disk D^{n+1} whose boundary is S^n is defined by $D^{n+1} = \{ x \in \mathbb{R}^{n+1} \mid ||x|| \leq 1 \}$. $\Diamond$

Proposition A.2. *For $n \geq -1$, both S^n and D^{n+1} are ωT_0.*

Our characterization will assign connectedness properties to CW complexes, of which there are two competing notions.

Definition A.3. A topological space X is *connected* if it cannot be expressed as a union of disjoint non-empty open sets. The space X is *locally connected* if every neighborhood of a point $x \in X$ contains a connected subneighborhood.

A topological space X is *path connected* if between any two points $x, y \in X$, there is a path $p : \mathbb{I} \to X$ s.t. $p(0) = x$ and $p(1) = y$. Similarly, X is *locally path connected* if every neighborhood of a point $x \in X$ contains a path connected subneighborhood. $\Diamond$

A.2 CW Complexes

Definition A.4. Let $n \geq 0$. Let A be a space, and J be some indexing set. If we have a map $[f_j]_{j\in J} : \coprod \partial D^n \to A$, then the pushout

$$\begin{array}{ccc} \coprod_{j\in J} \partial D^n & \xrightarrow[{[f_j]_{j\in J}}]{} & A \\ \big\downarrow & & \big\downarrow \\ \coprod_{j\in J} D^n & \longrightarrow & X \end{array}$$

is the *space obtained by attaching n-cells to A via* $[f_j]_{j\in J}$. We call $[f_j]_{j\in J}$ the *simultaneous attaching map.* Since the pushout is defined as a quotient of coproducts, we have a quotient map $q : A \coprod_{j\in J} D^n \to X$. ◊

A first observation is that A can be seen as a closed subset of X, while the interior of the disks can be seen as open subsets.

Proposition A.5 ([Geo08, Proposition 1.2.1.]). *Let X be obtained by attaching n-cells to A via $[f_j]_{j\in J}$. Then $q|_A : A \to X$ is a closed embedding, and $q|_{\mathring{(D^n_j)}} : \mathring{(D^n)} \to X$ is an open embedding.*

We then define some terminology to be able to refer to cells individually.

Definition A.6. Let X be obtained by attaching n-cells to A via $[f_j]_{j\in J}$. The *j-th n-cell* of X is the set $e^n_j = q(D^n_j)$, and f_j is the attaching map for e^n_j. Its associated *open cell* is the set $\mathring{e}^n_j = e^n_j - A$, which is open in X. Its associated *boundary* is the set $\dot{e}^n_j = e^n_j \cap A$. The map of pairs

$$\chi_j = q|_{D^n_j} : (D^n_j, \partial D^n_j) \to (e^n_j, \dot{e}^n_j)$$

is called the *characteristic map* for e^n_j. ◊

The notation suggests $\mathring{e}^n_j$ as the interior and $\dot{e}^n_j$ as the boundary of e^n_j. This is intuitive, except when $n = 0$. In this case $e^n_j = \mathring{e}^n_j$ is a singleton and $\dot{e}^n_j = \emptyset$, because the singleton e^n_j is open in X.

Now, a CW complex is an iterated (possibly infinite) attachment of 0-cells, then 1-cells, etc.

Definition A.7. A *CW complex* X is a space X and a sequence $\{X_n\}_{n\geq 0}$ such that

1. X_0 is obtained from $X_{-1} = \emptyset$ by attaching 0-cells.

2. X_{n+1} is obtained from X_n by attaching $n+1$-cells.

3. X is the colimit of the sequence of closed embeddings

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow ...$$

$\Diamond$

In practice, being a colimit means that X has the *weak topology* with respect to its skeletons X_n:

Proposition A.8. *Let X be a CW-complex. Then X is homeomorphic to the space $\bigcup_n X_n$ where $U \subseteq \bigcup_n X_n$ is open (resp. closed) iff $U \bigcap X_n$ is open (resp. closed) in X_n for all n.*

In particular, X_n is a closed subspace of X.

Proof. This is evident since colimits of topological spaces are quotients. An induction argument shows that $X_n \cap X_k$ for all k is closed in X_k, so X_n is closed in X. $\square$

Definition A.9. The n-cells of X are the n-cells of (X_n, X_{n-1}), i.e. $e_j^n = q^n(D_j^n)$ where $q^n : X_{n-1} \coprod_{j \in J_n} D_j^n \to X_n$ is the quotient map for the cell attachment (X_n, X_{n-1}). The associated open cell/interior $\mathring{e}_j^n$ and boundary $\dot{e}_j^n$ are defined appropriately as before. $\Diamond$

A.3 A countable CW complex with No Countable Basis

Which CW complexes lie in ωT_0? An initial conjecture would be that the countable CW complexes are precisely those with a countable basis. This is not true, by considering the following counter-example. The quotient space $\mathbb{R}/\mathbb{Z}$ may be constructed as the CW-complex with one 0-cell and $\mathbb{Z}$-many 1-cells. This is also called the *countably infinite rose*, since it looks like a rose with countably many petals.

We explain why the space $\mathbb{R}/\mathbb{Z}$ does not have a countable basis. Consider the quotient point $z \in \mathbb{R}/\mathbb{Z}$, the equivalence class of all the integers. We show that it does not have a countable local basis. That is, for every countable set $\{B_i\}_{i \in \mathbb{N}}$ of open neighborhoods of z, there is an open neighborhood that contains none of the B_i (i.e. no countable set of neighborhoods can act as a local basis). Note that this ensures it does not have a countable (global) basis. So, suppose we have such a $\{B_i\}_{i\in\mathbb{N}}$. Then by definition of the quotient topology, the inverse image of each B_i must contain some basic open neighborhood of each of the $k \in \mathbb{Z}$. Therefore, applying this fact with $k = i$, each B_i contains some non-integer $r_i > i$. Now, the set $\{r_i\}_{i\in\mathbb{N}}$ is closed since each r_i lives on a different petal so are far apart. Hence, its complement is open and moreover contains z. However, by construction it is missing an element of each B_i, showing that $\{B_i\}_{i\in\mathbb{N}}$ is not a local basis.

A.4 Characterization of the CW complexes in ωT_0

The essence of the problem in the counterexample beforehand lies in the fact that the central 0-cell z touches infinitely many 1-cells, allowing us to find one point per petal. In the case of a finite rose CW complex, we cannot make this argument since we run out of petals. Therefore, we need to impose the condition that cells only intersect finitely many other cells. This condition is called local finiteness.

Definition A.10 ([Geo08, pg. 221])**.** A CW-complex X is *locally finite* if each of its cells only intersects finitely many cells. $\diamondsuit$

While this gives us an intuitive condition, its not really a "local condition" in the sense of being a statement about neighborhoods of some point. Here, we show that the condition indeed can be stated in the form of local compactness.

Definition A.11 ([Hat02, Appendix A])**.** Let X be a CW complex, and $A \subseteq X$. We will describe an open neighborhood $N_\epsilon(A)$ of A where ϵ assigns a real number $0 < \epsilon_j^n \leq 1$ to each n-cell e_j^n. The construction is inductive by constructing open sets $N_\epsilon^n(A)$ in X_n, and then taking $N_\epsilon(A) = \bigcup_n N_\epsilon^n(A)$.

In the base case, let $N_\epsilon^0(A) = A \cap X_0$.

Suppose $N_\epsilon^n(A)$ has been defined. Then for each cell e_j^{n+1}, we construct the open set

$$U_j \cup V_j \subseteq D_j^{n+1}$$

where

- U is the ϵ_j-open ball around $\chi_j^{-1}(A) - \partial D^{n+1}$.

- V is the product $(1-\epsilon_j, 1] \times \chi_j^{-1}(N_\epsilon^n(A))$ with respect to the description of D^{n+1} via spherical coordinates $(r, s) \in [0, 1] \times \partial D^{n+1}$.

Then, we define

$$N^{n+1} = q_k(N^n(A) \coprod_j (U_j \cup V_j))$$

where $q_k : X^n \coprod_j D_j^{n+1} \to X^{n+1}$ is the quotient map. $\diamondsuit$

Lemma A.12 ([Hat02, Proposition A.3])**.** *Let X be a CW-complex. For disjoint closed sets A and B in X, we can find ϵ such that $N_\epsilon(A)$ and $N_\epsilon(B)$ are disjoint.*

Proposition A.13 ([Geo08, Prop 10.1.8.])**.** *Let X be a CW complex. Then the following conditions are equivalent:*

1. *X is locally finite.*
2. *X is neighborhood finite: each $x \in X$ has a neighborhood that intersects only finitely many cells of X.*
3. *X is locally compact.*

Proof. (1. $\implies$ 2.) Suppose X is locally finite, and consider an arbitrary $x \in X$. We will construct an open neighborhood of x that intersects only the same cells as x. Such a neighborhood concludes the proof as x must lie in some cell which by assumption intersects only finitely many cells, so also x must only intersect finitely many cells.

Now, there is a unique $k \geq 0$ such that $x \in X_k$ but $x \notin X_{k-1}$. Since x was not already added in X_{k-1}, it must lie in the interior $\mathring{e}_j^k$ of some freshly added cell. Now, consider the cells that do not intersect e_j^k. They form a finite subcomplex Y of X. Applying the previous lemma with $A = \{x\}$ and $B = Y$, we can find ϵ_Y such that $N_{\epsilon_Y}(x)$ is disjoint from $N_{\epsilon_Y}(Y)$. Now, by local finiteness, only finitely many cells intersect e_j^k but do not intersect x - let these be $e_1 \dots e_m$. We will apply the previous lemma with $A = \{x\}$ and $B = e_i$ for each $i \in \{1 \dots m\}$, leading to finitely many $\epsilon_1 \dots \epsilon_m$. Then our desired neighborhood is $N_\epsilon(x)$ where $\epsilon = \min(\epsilon_Y, \epsilon_1, \dots, \epsilon_m)$. Now $N_\epsilon(x)$ is disjoint from Y, so in particular is disjoint from any cell not intersecting e_j^k, and also is disjoint from any cell intersecting e_j^k but not x. That is, it only intersects cells that contain x.

(2. $\implies$ 3.) Let $x \in X$ with a neighborhood U that intersects only finitely many cells. Consider the closure S of U. Since U intersects only finitely

many cells, so does S. This means that S must lie in a finite union of cells, i.e. a compact subcomplex of X. Since S is a closed set, it must also be compact. This yields S as the desired compact neighborhood.

(2. $\implies$ 1.) Suppose every x has a neighborhood intersecting only finitely many cells. Consider a cell e in X. Now, e is compact, and to every $x \in e$ we can associate an open set U_x intersecting only finitely many cells by assumption. The collection $\{U_x\}_{x\in e}$ forms an open cover of e, so by compactness we have a finite subcover. The union of this finite subcover then will also only intersect finitely many cells, which means e can only intersect finitely many cells.

(3. $\implies$ 1.) Suppose X is locally compact. Taking an arbitrary cell e in X, suppose for contradiction that it intersects infinitely many cells. By local compactness, each point x in e has a compact neighborhood N_x with associated open subset $x \in U_x$, so $\{U_x\}_{x\in e}$ forms an open cover of e. However, the cell e is a finite subcomplex of X, so e is compact. That means from $\{U_x\}_{x\in e}$ we obtain a finite subcover, and the union of this finite subcover produces an open set U containing e, and the corresponding finite union of compact neighborhoods produces a compact neighborhood N of e.

Now, U intersects infinitely many cells $\{e_\alpha\}_{\alpha\in\mathcal{A}}$, but since it is open it must intersect the interior of the cells (or else it won't be open, since U is open iff $U \cap e$ is open in each cell e). For each e_α then, we can take an element $x_\alpha \in \mathring{(e_\alpha)} \cap U$. If we consider the set $D = \{\, x_\alpha \mid \alpha \in \mathcal{A} \,\}$, then we see that $D \cap e_\alpha$ is the singleton $\{x_\alpha\}$ which is closed in e_α, while $D \cap e$ for any other cell e is empty. Hence, D is closed in X, and therefore closed in N. A similar argument applies for any subset of D, showing that D is an infinite discrete (therefore not compact) subspace of N. On the other hand, D is a closed subspace of N, so it should be compact. This is a contradiction. $\square$

First, we observe that if X is locally finite then this almost gives us countability, except that X_0 is allowed to be uncountable.

Lemma A.14 ([Geo08, Exercise 3, Section 10.1])**.** *Let X be a CW complex. If X is not locally finite, then it does not have a countable local basis.*

Proof. Suppose X is not locally finite. Then by Proposition A.13, there is a point $x \in X$ such that all its neighborhoods intersect infinitely many cells. We will adapt the proof that the countable rose does not have a countable local basis.

Consider a countable set $\{B_i\}_{i \in \mathbb{N}}$ of open neighborhoods of x. Since each one intersects infinitely many cells, then for each i we can pick a point x_i in the intersection of B_i and the interior of some other cell $\mathring{e}_i$, such that $e_i \neq e_j$ for $i \neq j$. Now, the set $\{ x_i \mid i \in \mathbb{N} \}$ is closed since each x_i lies in the interior of a different cell. Hence, its complement is an open neighborhood of x which by construction contains none of the B_i. □

Lemma A.15 ([Geo08, Prop 10.1.25]). *Let X be a CW complex. If X is path connected and locally finite, then X has countable basis.*

Theorem A.16. *A CW-complex X has a countable local basis iff it is locally finite.*

Proof. ($\Longrightarrow$) By Lemma A.14.

($\Longleftarrow$) CW-complexes are locally path connected [Hat02, Proposition A.4.]. Hence, as per nlab, its path components are the same as its connected components, and X is homeomorphic to the disjoint sum of its connected components. Since disjoint sums reflect local finiteness, it suffices to show each of the connected component has a locally countable basis. Then, Lemma A.15 establishes that a path connected locally finite CW-complex has a countable basis, and therefore countable local basis. □

Theorem A.17. *A CW-complex X has a countable basis iff it is locally finite and has countably many connected components.*

Proof. ($\Longrightarrow$) Suppose X has a countable basis. Then it has a countable local basis, so by the previous theorem it is locally finite. Additionally, if it has uncountably many components then it has an uncountable basis (this is only true because X is a CW complex, so locally connected), so X must have countably many connected components.

($\Longleftarrow$) As before, we decompose X into its (countably-many) path components. Then it suffices to show that each component has a countable basis. This follows by Lemma A.15. □

Appendix B

Non-existence of Moore Path Spaces in Equ

In this appendix chapter, we define Moore structures, as well as demonstrate why we cannot find a Moore structure MX in Equ whose underlying elements are *continuous maps* $\mathbb{I}^n \to X$.

Definition B.1. Let X be an equilogical space. A *naive path* or $\mathbf{I}$-*path* is an equivariant continuous function $[0, 1] \to X$. A *naive Moore path* of length $r \in \mathbb{R}^+$ is an equivariant continuous function $[0, r] \to X$. Slightly abusing terminology, we will also refer to the equivalence class of these maps in this way. ♢

B.1 Moore Structures

We briefly introduce the relevant components of the Moore structure axioms. The canonical example to keep in mind the is the Moore path space from chapter 5.

Definition B.2. Let $\mathcal{C}$ be a finitely complete category $\mathcal{C}$. Then a Moore structure on $\mathcal{C}$ consists of

1. A pullback-preserving *functorial cocylinder* $M : \mathcal{C} \to \mathcal{C}$, with natural transformations

   $$s, t : M \Rightarrow id_{\mathcal{C}} \qquad r : id_{\mathcal{C}} \Rightarrow M \qquad \mu : M_t {\times}_s M \Rightarrow M \qquad \tau : M \Rightarrow M$$

   such that

a) $(X, MX, s_X, t_X, r_X, \mu_X)$ is an internal category for each $X \in \mathcal{C}_0$;

b) and τ is an identity-on-objects involution on this internal category.

2. M has *strength*, i.e. a natural transformation

$$\alpha_{X,Y} : MX \times Y \to M(X \times Y)$$

with respect to which s, t, r, μ, τ interact well.

3. There is a path contraction map $\eta : M \Rightarrow MM$ which also interacts well with the strength.

$\Diamond$

The intuition for $\alpha_{X,Y}$ is that it maps a path p on X and an element $y \in Y$ to the path in $X \times Y$ which is p on the left component and the constant path on the right component, with the same length as p. As a special case, we can view $M1$ as an object of path lengths, and $\alpha_{1,Y}$ constructs the constant path on $y \in Y$ with the specified length. By pullback preservation, the $\alpha_{1,Y}$ determine all other $\alpha_{X,Y}$. For example, in chapter 5, the Moore path space $M1$ is homeomorphic to $\mathbb{R}^+$, so elements of $M1$ correspond to length specifications.

In fact, from a strict interval object $(I, \bot, \top, \tau, \kappa)$ with a contraction map η, we can always induce such a Moore structure $(-)^I$ [BG12]. The contraction map on I induces the path contraction on $(-)^I$. However, this Moore structure is rather degenerate in that $1^I \cong 1$, so there is no meaningful notion of path length and path length object. The intuition for path contraction follows from this observation, although in the general case one has to factor in length considerations.

We refer to [Doc14] for further details and exposition on the precise axioms for a Moore structure.

B.2 Moore Structures on Equ

Assuming the elements are to be continuous $\mathbb{I}^n$-paths, there are some basic ideas to try in constructing the Moore structure, but they all fail due to a tension between path contraction and pullback preservation.

We begin by trying to adapt the Moore structure that we used to construct the model structure in QCB. First, we try one where we specify lengths as number of copies of $\mathbb{I}$.

Attempt 1

Definition B.3. The *First Moore structure* on an equilogical space Y is the pullback

$$\begin{array}{ccc} M_1 Y & \xrightarrow{\langle p, \mathsf{length} \rangle} & Y^{\mathbb{I}^\infty} \times \mathbb{N} \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{shift}} \\ Y & \xrightarrow[\mathsf{const}]{} & Y^{\mathbb{I}^\infty} \end{array}$$

Here, $\mathbb{I}^\infty$ is an infinite join of intervals $\mathbb{I} \vee \mathbb{I} \vee \ldots$. ◊

With some effort, one can find the following simpler presentation in Equ, which makes it clear that M_1 is functorial and pullback-preserving.

Lemma B.4. *The Moore path space $M_1 Y$ is isomorphic in* Equ *to the coproduct*

$$\coprod_{n \in \mathbb{N}} Y^{\mathbb{I}_n}.$$

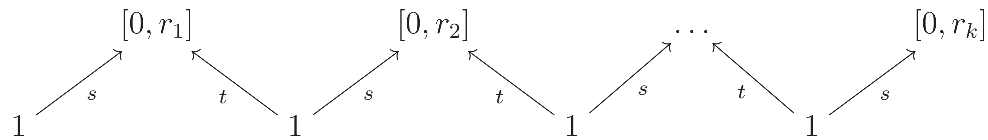
There is an obvious path contraction operation, however it is not equivariant since it requires us to have an equivalence between a length-$n+1$ path and a length-n path in Equ. This is not possible for M_1, because we defined it as disjoint sums between spaces of length-n paths, so there is zero interaction between paths of different lengths. We refer to the picture below for $M_{1.1}$ which has the same problem, but is perhaps more visually intuitive.

Attempt 1.1

Instead of using joins of unit-length $\mathbf{I}$, we may also consider joins of intervals of length $r \in \mathbb{R}^+$. This does not solve the problem of defining path contraction that we found in Attempt 1, but the "solution" of quotienting by constant paths works better in this context, which we explore in Attempt 2.

Let us denote the set of *non-empty* lists of non-zero real numbers by $\mathsf{List}_+(\mathbb{R}^+)$.

Definition B.5. Let X be an equilogical space, and $rs \in \mathsf{List}_+(\mathbb{R}^+)$. Let $\mathbb{I}^{rs}$ be the colimit in Equ of the diagram



where r_i is the i-th element of rs. A *Moore path* in X of length-data rs is a continuous equivariant function $\mathbb{I}^{rs} \to X$. ♢

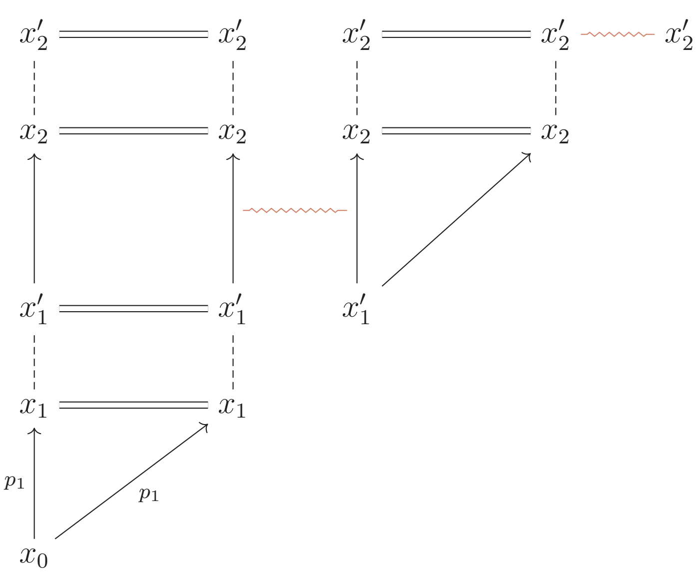
Definition B.6. The corresponding equilogical space of Moore paths over Y is defined by the coproduct

$$M_{1.1}Y := \coprod_{rs \in \mathsf{List}_+(\mathbb{R}^+)} Y^{\mathbb{I}^{rs}}$$

♢

Remark B.7. This was inspired by the construction of Moore structure for Equilogical spaces in [Vri15]. However, we use a finer topology here. ⊚

The problem of path contraction is still not solved since we do not identify paths of different length-data, so the obvious way of contracting paths is still not equivariant. With the direction of contraction being from left-to-right, the following picture explains the problem succintly when we try to contract a path, we eventually have to make an "equivalence jump", i.e. the red zig-zags, but these are missing in our definition of $M_{1.1}$.



Attempt 2

To solve the path contraction problem, we have to identify paths f of length-data $[0] +\!\!+\, xs$ with the corresponding path $f|_{I^{xs}}$ of length-data xs. This is where the difference between Attempt 1 and Attempt 1.1 comes in, since to fix Attempt 1 we would have to identify paths by the removal of constant naive paths. However, to fix the problem in Attempt 1.1, we only need to identify by removal of naive Moore paths of length $0 \in \mathbb{R}^+$ (which are always constant). The former does not seem to lead to a pullback preserving functor, while for the latter there is more hope.

Definition B.8. We define a reduction relation on $\mathsf{List}_+(\mathbb{R}^+)$ defined by

$$xs +\!\!+\, [0] +\!\!+\, ys \rightarrowtriangle xs +\!\!+\, ys$$

with $0 \notin xs$ and $xs \neq []$ or $ys \neq []$. This extends to a reduction relation on $|M_{1.1}Y|$ with

$$\frac{xs +\!\!+\, [0] +\!\!+\, ys \rightarrowtriangle xs +\!\!+\, ys}{(p : \mathbb{I}^{xs +\!\!+\, [0] +\!\!+\, ys} \to Y) \rightarrowtriangle (p|_{xs +\!\!+\, ys} : \mathbb{I}^{xs +\!\!+\, ys} \to Y)}$$

◊

Remark B.9. Clearly, $\rightarrowtriangle$ is a function on $\mathsf{List}_+(\mathbb{R}^+)$ and also normalizing, so it normalizes to a unique normal form which is either a non-empty list containing no 0s (for a list containing non-zeros), or a list containing just one 0 (for a list of zeroes). This means that $\rightarrowtriangle$ also normalizes, and we shall denote the $\rightarrowtriangle$-normal form of a Moore path p as $\mathsf{nf}(p)$. ⊚

Definition B.10. The modified Moore path space is now defined by further quotienting $M_{1.1}Y$ with the previously defined reducibility relation

$$M_2Y := (|M_{1.1}Y|, \sim_{M_{1.1}Y} \vee \rightarrowtriangle)$$

◊

Since $\sim_{M_2}$ as the equivalence closure of the join of two relations, we first give an easier-to-check condition for $\sim_{M_2}$.

Lemma B.11. *Let $p : \mathbb{I}^{xs} \to Y$ and $q : \mathbb{I}^{ys} \to Y$. Then*

$$\begin{aligned} p \sim_{M_2Y} q &\iff \exists h, h' : \mathbb{I}^{zs} \to Y . p \rightarrowtriangle^* h \sim_{M_{1.1}Y} h' \;{}^*\!\!\leftarrow q \\ &\iff \mathsf{nf}(p) \sim_{M_{1.1}Y} \mathsf{nf}(q) \end{aligned}$$

Proof. First, we note that $p \sim_{M_2Y} q$ iff there is a sequence of relations $\sim_{M_{1.1}Y}$, $\rightarrowtriangle$ and $\leftarrowtriangle$ starting in p and ending in q. To transform this sequence the first form given in this lemma, we simply have to prove that $p \sim_{M_{1.1}Y}\rightarrowtriangle q$ implies $p \rightarrowtriangle\sim_{M_{1.1}Y} q$ (then it dually follows that $\leftarrowtriangle\sim_{M_{1.1}Y}$ implies $\sim_{M_{1.1}Y}\leftarrowtriangle$), and that $p \leftarrowtriangle\rightarrowtriangle q$ implies $p = q$.

So, suppose $p \sim_{M_{1.1}Y} q' \rightarrowtriangle q$. Then $p, q' : \mathbb{I}^{xs +\!\!+\ [0] +\!\!+\ ys} \to Y$ and $q = q'|_{xs +\!\!+\ ys} : \mathbb{I}^{xs +\!\!+\ ys} \to Y$. Define $p' := p|_{xs +\!\!+\ ys}$. Then $p \rightarrowtriangle p'$, and $p \sim_{M_{1.1}Y} q'$ implies $p' = p|_{xs +\!\!+\ ys} \sim_{M_{1.1}Y} q'|_{xs +\!\!+\ ys} = q$.

Next, if $p \leftarrowtriangle q' \rightarrowtriangle q$ then $p = q'|_{xs++ys} = q$.

For the second form, we have to prove that $\rightarrowtriangle$ preserves $\sim_{M_{1.1}Y}$, in the sense that if $h \sim_{M_{1.1}Y} h'$, $h \rightarrowtriangle g$ and $h' \rightarrowtriangle g'$ then $g \sim_{M_{1.1}Y} g'$. But this is already clear by the same considerations as for the first form. □

With this, we can easily prove functoriality and preservation of pullbacks.

Corollary B.12. *$M_2(-)$ extends to a functor $M_2 : \mathsf{Equ} \to \mathsf{Equ}$.*

Proof. Given a map $[f] : Y \to Z$, we define $M_2[f]$ as the equivariance class of the map

$$(p : \mathbb{I}^{xs} \to Y) \mapsto (fp : \mathbb{I}^{xs} \to Z)$$

First, this definition is clearly independent of the choice of f since $f \sim f'$ implies $fp \sim_{M_{1.1}Z} f'p$ and therefore $fp \sim_{M_2Z} f'p$. Second, this definition gives an equivariant map since $p \rightarrowtriangle p'$ implies $fp \rightarrowtriangle fp'$, and so

$$p \rightarrowtriangle^* h \sim_{M_{1.1}Y} h' \ {}^*\!\!\leftarrowtriangle q$$

implies

$$fp \rightarrowtriangle^* fh \sim_{M_{1.1}Y} fh' \ {}^*\!\!\leftarrowtriangle fq.$$

□

Corollary B.13. *M_2 preserves pullbacks.*

Proof. Let $Y \overset{\pi_1}{\leftarrow} P \overset{\pi_2}{\rightarrow} X$ be the standard pullback of $Y \overset{f}{\rightarrow} Z \overset{g}{\leftarrow} X$. Let Q be the pullback of $M_2Y \overset{M_2f}{\rightarrow} M_2Z \overset{M_2g}{\leftarrow} M_2X$. Then there is an induced map $h : M_2P \to Q$, which (is represented by the function that) sends a path $p : \mathbb{I}^{xs} \to P$ to two paths $(\pi_1 p, \pi_2 q)$. We must now define an up-to-equivalence inverse h^{-1} of this induced map. Hence, define

$$h^{-1} : (p_1 : \mathbb{I}^{xs} \to Y, p_2 : \mathbb{I}^{ys} \to X) \mapsto \langle \mathsf{nf}(p_1), \mathsf{nf}(p_2) \rangle : \mathbb{I}^{zs} \to P$$

First of all, this map is well-defined since by our previous characterization of $\sim_{M_2 Z}$, $fp_1 \sim_{M_2 Z} gp_2$ implies $f\mathsf{nf}(p_1) = \mathsf{nf}(fp_1) \sim_{M_{1.1} Z} \mathsf{nf}(gp_2) = g\mathsf{nf}(p_2)$, so $\langle \mathsf{nf}(p_1), \mathsf{nf}(p_2) \rangle$ indeed maps into the pullback. One can also see that this map is equivariant by similar considerations.

We also have that $h^{-1}h$ normalizes the given path p:

$$h^{-1}h(p) = \langle \pi_1 \mathsf{nf}(p), \pi_2 \mathsf{nf}(p) \rangle = \mathsf{nf}(p) \sim_{M_2 P} p$$

while $hh^{-1}(p_1, p_2) = h \langle \mathsf{nf}(p_1), \mathsf{nf}(p_2) \rangle = (\mathsf{nf}(p_1), \mathsf{nf}(p_2)) \sim_Q (p_1, p_2)$.

We now prove that the underlying function of h^{-1} is continuous. Suppose then that we have a subbasic-open neighborhood $\langle \mathsf{nf}(p_1), \mathsf{nf}(p_2) \rangle \in \mathsf{M}(K, V) \subseteq [\mathbb{I}^{zs} \to_{\mathsf{Equ}} P]$. Since V is induced by an open set V' in $Y \times X$, we can decompose it as a union of products $V' = \bigcup_{i \in I} U_i \times V_i$. Then K corresponds by inclusion to a compact set in $\mathbb{I}^{ys}$ and $\mathbb{I}^{xs}$ respectively, so we have the open neighborhood

$$(p_1, p_2) \in (\bigcup_{i \in I} M(K, U_i) \times M(K, V_i)) \cap |Q|.$$

Moreover, $h^{-1}[(\bigcup_{i \in I} M(K, U_i) \times M(K, V_i)) \cap |Q|] \subseteq \mathsf{M}(K, V)$. To prove this, take any $i \in I$ with an element $(p_1', p_2') \in M(K, U_i) \times M(K, V_i) \cap |Q|$. Then we have $\langle \mathsf{nf}(p_1), \mathsf{nf}(p_2) \rangle [K] \subseteq U_i \times V_i$ because the normalization only removes the 0-length naive Moore intervals, while K is a subset of the non-zero naive Moore intervals. Therefore, $\langle \mathsf{nf}(p_1), \mathsf{nf}(p_2) \rangle \in \mathsf{M}(K, \bigcup_{i \in I} U_i \times V_i)$. □

Remark B.14. Note that the proofs above rely on many properties that are only true because we only quotient by 0-length Moore intervals, which is only possible by modifying Attempt 1.1, and not Attempt 1. ◎

Unfortunately, the path space $M_2 Y$ still doesn't work for defining path contraction. This time, the problem is with continuity. In order to define the underlying function of the path contraction operation $\Gamma_Y : M_2 Y \to M_2 M_2 Y$, it suffices to define a truncation map

$$\begin{aligned}
\Gamma_Y' &: [\mathbb{I}^{xs} \to_{\mathsf{Equ}} Y] \times \mathbb{I}^{xs} \to \coprod_{xs \in \mathsf{List}_+(\mathbb{R}^+)} [\mathbb{I}^{xs} \to_{\mathsf{Equ}} Y] \\
\Gamma_Y' &: (p, (i, t)) \mapsto ((xs[i] - t) : xs[i+1 :], \\
&\qquad (j, s) \mapsto \begin{cases} p(i + j, t + s) & j = 0 \\ p(i + j, s) & j > 0 \end{cases}
\end{aligned}$$

Unfortunately, since the codomain has the coproduct topology, we need to find an open neighborhood U around $(i,t) \in [0, xs[i]]$ such that truncating by any element of U results in the length data $(xs[i]-t) +\!\!+\ xs[i+1:]$. This is not possible unless U is the singleton $\{(i,t)\}$, but this singleton is not open. The problem then is that we do not incorporate the topology of $\mathbb{R}^+$ in M_2. This is best illustrated when considering the "object of path lengths" $M_2 1$, we see that $M_2 1$ is the space

$$(\mathsf{List}_+(\mathbb{R}^+), \leftarrow\!\!\rightarrow^*)$$

with the discrete topology.

Attempts 3 & 4

Unfortunately, the coproduct topology we gave to the underlying space of M_2 is too discrete, making the path contraction discontinuous. Indeed, this is illustrated most clearly by $M_2 1$ having the discrete topology. But it should really incorporate the topology of $\mathbb{R}^+$, i.e. it should have the topology of $\coprod_{n>0}(\mathbb{R}^+)^n$. We made two attempts to retopologize M_2. In the first attempt, we followed the topology in [Vri15], but this breaks pullback preservation, even though it makes path contraction work again. In the second attempt, we made a minor modification to the topology we use on $\mathbb{R}^+$ in order to make $\{ 0 \}$ closed in $\mathbb{R}^+$, which again fixes pullback preservation but breaks path contraction.

This seemingly unresolveable tension suggests we need to rethink and reconceptualize the underlying elements of the Moore structure. An ongoing theme in this exploration is also the claim that pullback preservation does not hold without an explicit counterexample. However, we have not developed sufficient tools to answer when two equilogical spaces are non-isomorphic in Equ, so we cannot decisively prove such claims. In chapter 6, I claim that doing so requires a more homotopical understanding of the realizability structure in Equ itself.